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Zusammenfassung

Wir betrachten das Ewens-Maß auf der symmetrischen Gruppe S_n bedingt auf das Ereignis, dass keine Zykel makroskopischer Länge auftreten. Dieses Wahrscheinlichkeitsmaß kann durch Zykelgewichte dargestellt werden, welche von der Größe des Systems n abhängen. Ziel ist es, das asymptotische Verhalten der sich ergebenden Zykelstruktur der zufälligen Permuationen ohne makroskopische Zykel im Limes großer n zu beschreiben. Wir zeigen zunächst, dass die gemeinsame Verteilung der Anzahlen in einem präzisen Sinne kurzer Zykel asymptotisch durch die Bedingung nicht beeinflusst wird und in Totalvariationsdistanz gegen unabhängige Poisson-verteilte Zufallsvariablen konvergiert. Aus dieser Tatsache ergibt sich zudem, dass kumulative Anzahlen von Zykeln kurzer Länge denselben funktionalen Grenzwertsatz erfüllen wie unter dem klassischen Ewens-Maß. Im Folgenden werden Grenzwertsätze für die gemeinsame Verteilung von Anzahlen von Zykeln gegebener Länge bewiesen. Der Grenzwert hängt hierbei stark von der konkret gewählten Bedingung und der betrachteten Zykellänge ab. In einem nächsten Schritt stellen wir einen zentralen Grenzwertsatz für die Gesamtzahl der Zykel vor, woraufhin wir das Verhalten kumulativer Zykel- und Indexanzahlen betrachten. Für diese beweisen wir jeweils die Existenz einer Grenzgestalt sowie einen funktionalen Grenzwertsatz für die Fluktuationen um diese Grenzgestalt, wobei die Fluktuationen gegen die Brownsche Brücke konvergieren. Aus den Grenzgestalten können wir zudem Schlüsse über das asymptotische Verhalten eines typischen Zykels ziehen. Des Weiteren bestimmen wir das Verhalten der längsten Zykel und zeigen in diesem Zusammenhang in einem bestimmten Regime Konvergenz kumulativer Zykelanzahlen gegen einen Poisson-Prozess.

Abstract

We consider the Ewens measure on the symmetric group S_n conditioned on the event that no cycles of macroscopic lengths occur and investigate the resulting cycle structure of random permutations without macroscopic cycles in the limit $n \to \infty$. This probability measure can be represented by cycle weights which depend on the system size n. We first establish that the joint distribution of the cycle counts of short cycles is not affected by the conditioning and converges to independent Poisson-distributed random variables in total variation distance. Cumulative cycle numbers of short cycles hence fulfil the same functional central limit theorem as under the classical Ewens measure. Then limit theorems are proved for (the joint distribution of) general individual cycle numbers where the limit strongly depends on the concrete choice of constraint in the conditioning and the cycle lengths in question. Having examined properties related to individual cycle numbers, we turn to the total number of cycles which satisfies a central limit theorem. For cumulative cycle and index numbers we prove the existence of limit shapes and functional limit theorems for the fluctuations about these limit shapes, the limit of the fluctuations being the Brownian bridge. The limit shapes also allow us to determine the asymptotic behaviour of a typical cycle. Lastly, we present findings concerning the distribution of the longest cycles in the model and in this context show convergence of cumulative cycle numbers in a certain regime to a Poisson process.

Preface

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which are listed as [**9**] and [**10**] in the bibliography. More precise references will be provided in each section.
First and foremost, I wish to thank my thesis advisor Volker Betz for his constant support during the research for this thesis. His door has literally always been open and I especially appreciate his encouragement to also take interest in neighbouring fields both mathematical and physical.
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Contents

Introduction

Random permutations have been objects of mathematical research for many decades. In the classical and most famous model, the symmetric group S_n is endowed with the uniform measure $\mathbb{P}_n^{(1)}$, so each permutation of n indices has probability $\frac{1}{n!}$. Since each permutation can be written as a product of disjoint cycles, one often considers the cycle structure $(C_1(\sigma), C_2(\sigma), \ldots, C_n(\sigma))$ of a permutation σ, where $C_j(\sigma)$ denotes the number of cycles of length j in σ, and tries to prove statements about the distribution of said cycle structure, typically in the limit of large n. Strictly speaking, the cycle counts $C_j : S_n \to \mathbb{N}_0$ have different domains depending on n, but for the sake of brevity we suppress this fact in the notation. An approach focused on the cycle structure is sound when the probability of a permutation only depends on its cycle counts C_j. Random permutations with cycle weights are the prototype of probability measures satisfying this condition: They are defined by assigning the probability of $\mathbb{P}[\{\sigma\}] = \frac{1}{Z}\prod_{j=1}^{n} \vartheta_j^{C_j(\sigma)}$ to σ, where the $\vartheta_j \geq 0$ are fixed numbers called the cycle weights and Z is a normalizing constant. If one chooses $\vartheta_j = \vartheta > 0$ for all j, one obtains the Ewens measure $\mathbb{P}_n^{(\vartheta)}$ **[26]** which first arose in mathematical biology in the context of genetics. The uniform measure $\mathbb{P}_n^{(1)}$ is recaptured as the special case when $\vartheta = 1$. Classical results about the cycle structure under the Ewens measure include the convergence of the renormalized lengths of the longest cycles to a Poisson-Dirichlet distribution **[37, 59]**, convergence of the joint distribution of counts of non-macroscopic cycles to independent Poisson-distributed random variables in total variation distance **[4, 3]**, a central limit theorem for the total number of cycles $C = \sum_{j=1}^{n} C_j$ **[31, 58, 39]**, and a functional central limit theorem for cumulative cycle counts **[34, 18]**. In particular, almost all indices are contained in macroscopic cycles: For instance, under the uniform measure $\mathbb{P}_n^{(1)}$, for any $\epsilon > 0$ one can show that the probability of a fixed index belonging to a cycle of length less than ϵn converges to ϵ as n tends to infinity.

Many models with more general cycle weights than $\vartheta_j = \vartheta$ which are still fixed have also been studied (see, e.g., **[12, 25, 16]**). Models of special interest in the context of this thesis are those which restrict the set of allowed cycle lengths in a permutation. One such model goes by the name of random A-permutations (cf., e.g., **[66, 67, 68, 70]** and ample references provided therein): Here one fixes a set $A \subset \mathbb{N}$ of allowed cycle lengths and then defines the cycle weights as $\vartheta_j = 1$ if $j \in A$ and $\vartheta_j = 0$ if $j \notin A$. The set of permutations with non-zero weights is then referred to as $T_n(A)$. Among other things, if the constraint imposed is not too strong, i.e. if $|T_n(A)| \sim n^{\beta}(n-1)!$ for some $\beta > 0$, one shows a central limit theorem for the total number of cycles and the convergence of finite-dimensional distributions of cycle counts to independent Poisson-distributed random variables.

A next step is allowing the set A to depend on n. The paper **[50]** considers "intervals" $A_n = \{a_n + 1, a_n + 2, \ldots, n\}$ for $a_n = o(n)$, i.e. $\lim_{n\to\infty} a_n/n = 0$, which contain all macroscopic cycles. Features of the model include a functional central limit theorem with logarithmically growing expected total number of cycles and the approach in **[50]** can also deal with additionally introducing cycle weights ϑ_j.

It should be noted that the problem is generally much harder if one considers cycle weights which depend on the system size n. Yet, such models are of particular interest in mathematical physics since certain models of spatial random permutations arising from a background in quantum mechanics and statistical physics fall into this category (see, e.g., **[11, 33]** and **[8]** for further references and simulations). Further results have been obtained in **[15]** for a surrogate model defined by more explicit cycle weights intended to approximate relevant features of spatial random permutations. One interesting property in this regard is that, under suitable assumptions, the finite cycles in surrogate-spatial permutations contain a positive fraction of indices. For entropic reasons, such a behaviour is very rare in models of random permutations without relation to a spatial component, but the norm in spatial random permutations. Recently **[23]** harnessed techniques adapted to random permutations with cycle weights to treat spatial random permutations directly.

In this thesis we are going to consider a special instance of random A-permutations where the set A is allowed to depend on the system size n in a natural way and we further allow constant cycle weights $\vartheta > 0$. Complementary to **[50]**, we forbid the permutations from having cycles with lengths above a certain n-dependent threshold: Let $\alpha : \mathbb{N} \to \mathbb{N}$ be a sequence with $1 \leq \alpha(n) \leq n$

for all n and define the set

$$S_{n,\alpha} := \{\sigma \in S_n |\, \text{no cycle in } \sigma \text{ is longer than } \alpha(n)\}$$

of permutations without long cycles. Then, for $\vartheta > 0$, $\mathbb{P}_{n,\alpha}^{(\vartheta)}[\cdot] := \mathbb{P}_n^{(\vartheta)}[\cdot|\, S_{n,\alpha}]$ is a probability measure concentrated on $S_{n,\alpha}$ to which we refer as random permutations without macroscopic cycles if $\alpha(n) = o(n)$. Since $\mathbb{P}_n^{(\vartheta)}[S_{n,\alpha}] \xrightarrow{n\to\infty} 0$ if $\alpha(n) = o(n)$ (almost all indices belong to macroscopic cycles), this conditioning is singular in the limit. Depending on the value of ϑ, $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ is also called the constrained Ewens or uniform measure. Since $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ originates from conditioning the Ewens measure, we have

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}[\{\sigma\}] = \frac{1}{Z_{n,\alpha,\vartheta}} \frac{1}{n!} \vartheta^{C(\sigma)}$$

for $\sigma \in S_{n,\alpha}$, where

$$Z_{n,\alpha,\vartheta} = \frac{1}{n!} \sum_{\sigma \in S_{n,\alpha}} \vartheta^{C(\sigma)}$$

is a normalizing constant and the factor of $n!$ appears due to convention. In most cases we will restrict considerations to sequences α satisfying

$$n^{a_1} \le \alpha(n) \le n^{a_2}$$

for fixed numbers $a_1, a_2 \in (0,1)$ and for all $n \in \mathbb{N}$. As we will see, this algebraic growth condition ensures a certain uniformity of behaviour which is still rich enough to yield interesting results. Further motivation is provided by numerical studies in **[8]** which suggest that permutations whose longest cycles are of algebraic order occur naturally in two-dimensional spatial random permutations. Even though the model in **[8]** certainly entails no hard constraint on the maximal cycle length, random permutations without macroscopic cycles may be considered a relevant toy model in this respect.

We obtain the following main results: For the joint distribution of counts of short cycles we can show that in the limit of large n they behave as in the unconstrained case. More precisely, the cycle counts $\left(C_1, C_2, \ldots, C_{b(n)}\right)$ converge in total variation distance to independent Poisson-distributed random variables for any sequence b satisfying $b(n) = o(\alpha(n) / \log(n))$. For short cycles we thus also recover the classical functional central limit theorem for cumulative cycle counts. Concerning long cycles, we identify the scale $\alpha(n)(1 + \log(t) / \log(n/\alpha(n)))$, $t \in (0,1)$, at which almost all cycles and indices live and prove the existence of limit shapes for cumulative cycle numbers and cumulative index numbers as well as functional central limit theorems for the fluctuations about these limit shapes. It is striking in this respect that limit shapes and fluctuations of cycles and indices move exactly in parallel, which leads to the limit of the fluctuations being the Brownian bridge. Hence, in this scaling, the total number of cycles C becomes deterministic in the limit. Nevertheless, we are able to prove a central limit theorem for the total number of cycles which employs a different scaling.

Further results include limit theorems for joint distributions of individual cycle counts and an investigation of typical and longest cycles. It should be noted that the asymptotic behaviour of individual cycle counts strongly depends on the maximal cycle length $\alpha(n)$ of the system. The faster $\alpha(n)$ grows, the closer the model of random permutations without macroscopic cycles is to the Ewens measure. Hence, if $\alpha(n) = \mathcal{O}(\sqrt{n})$, i.e. there are constants $K, N > 0$ such that $\alpha(n) \le K\sqrt{n}$ for all $n \ge N$, the expected value $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{\alpha(n)}\right]$ of the number of cycles of maximal length is going to diverge, whereas we have $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{\alpha(n)}\right] \xrightarrow{n\to\infty} 0$ if $\alpha(n)$ grows faster than $n^{\frac{1}{2}+\delta}$ for some $\delta > 0$. Note that there is also an intermediate regime where $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{\alpha(n)}\right]$ stays bounded, but does not converge to zero.

The methods employed in the present thesis are grounded in analytic combinatorics and asymptotic analysis. Weak convergence of probability measures is generally established by proving convergence of relevant moment-generating functions. Analytic combinatorics provides us with a framework from which to deduce the structure of said moment-generating functions and we apply the saddle-point method in order to extract asymptotic information. Here we build on **[45]**, where Manstavičius and Petuchovas apply the saddle-point method to calculate the probability $\mathbb{P}_n^{(1)}[S_{n,\alpha}]$ for a wide range of sequences α. In particular, if $n^{a_1} \le \alpha(n) \le n^{a_2}$ for some $0 < a_1 < a_2 < 1$, they show that $\mathbb{P}_n^{(1)}[S_{n,\alpha}]$ decreases faster than any power of n^{-1}, which shows that the model of random permutations without macroscopic cycles is far removed from random A-permutations with a fixed set A as discussed above. Asymptotic analysis and extended calculations are then required to interpret the resulting formulas appropriately and we arrive at a fairly complete picture of random permutations without macroscopic cycles.

As to the content of the thesis, Section 1.1 presents a selection of properties of the cycle structure under the Ewens measure $\mathbb{P}_n^{(\vartheta)}$. Section 1.2 covers the connection between weak convergence of probability measures and convergence of moment-generating functions while Section 1.3 gives a concise introduction to analytic combinatorics and generating functions.

Having thus presented the background and certain preliminaries, we start discussing random permutations without macroscopic cycles. In a first step, Section 2.1 treats an instance of the saddle-point method suited to most needs in this thesis and for this purpose introduces the notion of admissibility. Section 2.2 then establishes that the model of random permutations without macroscopic cycles falls indeed within the scope of this framework, which paves the way for the following sections. Section 2.3 provides asymptotics of the expected values of individual cycle counts and thoroughly discusses the influence of the maximal cycle length $\alpha(n)$ in this context. These results are refined in Section 2.4 which deals with limit distributions of individual cycle numbers and shows that these are essentially determined by the behaviour of the respective asymptotic expected values. Section 2.5 then presents the first theorem which states that the cycle structure of short cycles converges in total variation distance to independent Poisson-distributed random variables. Further conclusions are drawn which altogether justify the statement that the behaviour of counts of short cycles is asymptotically not affected by imposing the constraint of a maximal cycle length $\alpha(n)$. The following Section 2.6 establishes a central limit theorem for the total number of cycles and does so for a wide range of sequences α which need not grow algebraically in n. The theorems concerning limit shapes and the fluctuations about the limit shapes are given in Section 2.7 and cast a light on the asymptotic behaviour of long cyles. Sections 2.8 and 2.9 then deal with the distributions of longest and typical cycles, respectively, which includes proving convergence of cumulative counts of long cycles to a Poisson process in a certain regime.

In the third part of the thesis, we take a step back and provide an overview of results concerning other models of random permutations with cycle weights depending on the size n of the system in Section 3.1. Section 3.2 concludes the thesis by outlining a number of ways in which the model of random permutations without macroscopic cycles might be generalized.

CHAPTER 1

Background and Preliminaries

1.1. Classical Models: Uniform Random Permutations and the Ewens Measure

In the following, we will review a number of results about the classical models of uniform random permutations and the Ewens measure. Recall that the Ewens measure $\mathbb{P}_n^{(\vartheta)}$ for $\vartheta > 0$ on the symmetric group S_n assigns to each permutation $\sigma \in S_n$ the probability $\mathbb{P}_n^{(\vartheta)}[\{\sigma\}] = \frac{\vartheta^{C(\sigma)}}{Z_\vartheta(n)}$, where $C(\sigma)$ is the total number of cycles in σ and $Z_\vartheta(n)$ is a normalizing constant. When we speak of ϑ-biased random permutations, we also refer to the Ewens measure. The uniform measure can be considered as a special case of the Ewens measure when $\vartheta = 1$. An important reference for this section is [**2**]. Historically, the Ewens measure arose first in the field of population genetics (see [**26**]).

1.1.1. The Feller Coupling and the Total Number of Cycles. The Feller coupling dates back to [**27, 54**] and is an important tool for analyzing random permutations. It rests on the canonical cycle notation for permutations. We can write any $\sigma \in S_n$ as a product of disjoint cycles in the following unique way: The first cycle starts with the index 1 and its further elements are given by the definition of σ. If not all n indices are contained in the first cycle, we select the smallest index which is not contained in the first cycle and make it the starting point of the second cycle. This procedure can be iterated in such a way that the kth cycle starts with the smallest index which is not contained in the first $k-1$ cycles (if such an index exists). For instance, the canonical cycle notation of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 5 & 8 & 1 & 4 & 2 & 6 \end{pmatrix} \in S_8$$

is given by

$$\sigma = (\ 1 \quad 3 \quad 5\ )(\ 2 \quad 7\ )(\ 4 \quad 8 \quad 6\ ).$$

If we build up the canonical cycle notation of a permutation in S_n from the left, we always start with $(1$. We then have n possibilities to continue: Either we close the first cycle and start the second one, which leads to $(1)(2$, or we continue the first cycle for which we have the options of $(1 \quad 2$, $(1 \quad 3$, ..., and $(1 \quad n$. This procedure can be iterated so that in the kth step we can either close the cycle in question or continue it in $n-k$ different ways. Hence, there is a bijection

$$(D_1, D_2, \ldots, D_n) : \{1\} \times \{1,2\} \times \cdots \times \{1,2,\ldots,n\} \to S_n$$

where $D_{n-k+1} = 1$ if the cycle is closed in the k-th step and $D_{n-k+1} = l > 1$ if the cycle is continued with the $(l-1)$th smallest index not used in the $k-1$ previous steps. If we identify the sets $\{1\} \times \{1,2\} \times \cdots \times \{1,2,\ldots,n\}$ and S_n and consider the uniform measure $\mathbb{P}_n^{(1)}$ on S_n, one immediately sees that the D_k are independent and that D_k is uniformly distributed on $\{1,2,\ldots,k\}$ for each k. Observe also that the total number of cycles of a permutation σ can be written as

$$C(\sigma) = \sum_{k=1}^{n} \mathbb{1}_{\{D_k = 1\}}. \tag{1.1.1}$$

One can therefore show that, if we endow S_n with $\mathbb{P}_n^{(\vartheta)}$ for $\vartheta > 0$ instead of the uniform measure, the D_k are still independent. We further have

$$\mathbb{P}_n^{(\vartheta)}[D_k = 1] = \frac{\vartheta}{\vartheta + k - 1}$$

and

$$\mathbb{P}_n^{(\vartheta)}[D_k = l] = \frac{1}{\vartheta + k - 1}$$

for $2 \le l \le k$ and $1 \le k \le n$. Thus, the random variables $\mathbb{1}_{\{D_k=1\}}$ are independent and Bernoulli distributed with parameter $\frac{\vartheta}{\vartheta+k-1}$. If one applies the Lindeberg-Feller central limit theorem (cf., e.g., [**36**, Theorem 5.12]) to Equation (1.1.1), one concludes that, under $\mathbb{P}_n^{(\vartheta)}$,

$$\frac{C - \vartheta \log(n)}{\sqrt{\vartheta \log(n)}} \xrightarrow{d} X, \tag{1.1.2}$$

where X has the standard normal distribution. For $\vartheta = 1$, [**31, 58**] applied generating functions to prove Equation (1.1.2). The paper [**39**] gave another proof of the same statement.
Observe that the Feller coupling is of limited use for investigating random permutations without macroscopic cycles since the D_k are no longer independent in this case.

1.1.2. The Conditioning Relation and Non-Macroscopic Cycles. As we will see in Section 1.1.2.1, the law of the cycle counts of uniform and ϑ-biased random permutations is connected to the law of independent Poisson-distributed random variables conditioned on a certain set. It can further be shown that the joint law of non-macroscopic cycle counts converges in total variation distance to said independent Poisson-distributed random variables, which will be discussed in Section 1.1.2.2. Section 1.1.2.3 then presents a further consequence which is a functional central limit theorem for cumulative cycle counts. The methods employed throughout this section also apply to a wide range of combinatorial structures aside from permutations. An account of the corresponding general theory of logarithmic combinatorial structures can be found in [**2**].

1.1.2.1. *The Conditioning Relation.* We start with some general considerations concerning permutations. Let $N_{\boldsymbol{c}}$ for $\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{N}_0^n$ such that $n = \sum_{k=1}^n kc_k$ be the number of permutations in S_n whose cycle counts are given by $C_k = c_k$ for all $1 \le k \le n$. By Cauchy's formula (see [**2**, Equation (1.2)]), we obtain that

$$N_{\boldsymbol{c}} = \frac{n!}{\prod_{k=1}^n k^{c_k} c_k!}. \tag{1.1.3}$$

Cauchy's formula may be understood as follows: If we fix a cycle structure $\boldsymbol{c}$, we start with $n!$ possibilities of distributing the indices $1, 2, \ldots, n$ among the different cycles. That number has to be divided by a factor of $\prod_k c_k!$ since we can permute cycles of the same lengths without changing the permutation. The second correcting factor of $\prod_k k^{c_k}$ arises from the fact that each cycle is itself invariant under cyclic permutations. If we define $\boldsymbol{C} = (C_k)_{k=1}^n$, we therefore arrive at the Ewens Sampling Formula [**26**] which is given by

$$\mathbb{P}_n^{(\vartheta)}[\boldsymbol{C} = \boldsymbol{c}] = \frac{n!}{Z_\vartheta(n)} \mathbb{1}_{\{n=\sum_{k=1}^n kc_k\}} \prod_{k=1}^n \left(\frac{\vartheta}{k}\right)^{c_k} \frac{1}{c_k!}.$$

Let Z_k, $1 \le k \le n$, be independent Poisson-distributed random variables with parameters $\frac{\vartheta}{k}$ and define

$$T_{b_1 b_2} := \sum_{k=b_1+1}^{b_2} kZ_k. \tag{1.1.4}$$

Then one easily sees that

$$\mathbb{P}_n^{(\vartheta)}[\boldsymbol{C} = \boldsymbol{c}] = \mathbb{P}[\boldsymbol{Z} = \boldsymbol{c} | T_{0n} = n] \tag{1.1.5}$$

holds for all $\boldsymbol{c} \in \mathbb{N}_0^n$, which is the so-called conditioning relation and of central importance for the study of logarithmic combinatorial structures. From the point of view of statistical mechanics, the random variables Z_k play the role of the grandcanonical ensemble. Conditioning the cycles to contain the correct number of indices leads to the canonical ensemble which is in this case given by the Ewens measure on the set of permutations.
A logarithmic combinatorial structure is to be defined as a decomposable combinatorial structure whose component counts satisfy a conditioning relation such as in Equation (1.1.5), where the Z_k satisfy both $k\mathbb{P}[Z_k = 1] \to \vartheta$ and $k\mathbb{E}[Z_k] \to \vartheta$ as $k \to \infty$ for some $\vartheta > 0$. The book [**2**] develops the general theory of such structures and shows that many statements which hold for permutations can also be proved in the more general setting.

1.1.2.2. *Convergence in Total Variation Distance.* The conditioning relation given in Equation (1.1.5) is the starting point for investigating the joint law of the counts of non-macroscopic cycles. Recall that the total variation distance between two probability measures $\mathbb{P}$ and $\mathbb{P}'$ on a discrete space Ω is given by $\|\mathbb{P}-\mathbb{P}'\|_{\mathrm{TV}} = \sum_{\omega\in\Omega}\left(\mathbb{P}\left[\{\omega\}\right]-\mathbb{P}'\left[\{\omega\}\right]\right)_+$. Let further $\mathcal{L}(\cdot)$ denote the law of a random variable and write $\boldsymbol{Z}_{b(n)} = (Z_k)_{k=1}^{b(n)}$ and $\boldsymbol{C}_{b(n)} = (C_k)_{k=1}^{b(n)}$. Then, in the situation described in Section 1.1.2.1, we have

$$d_{b(n)} := \left\|\mathcal{L}\left(\boldsymbol{Z}_{b(n)}\right)-\mathcal{L}\left(\boldsymbol{C}_{b(n)}\right)\right\|_{\mathrm{TV}} = \sum_{\boldsymbol{c}\in\mathbb{N}^{b(n)}}\left(\mathbb{P}\left[\boldsymbol{Z}_{b(n)}=\boldsymbol{c}\right]-\mathbb{P}_n^{(\vartheta)}\left[\boldsymbol{C}_{b(n)}=\boldsymbol{c}\right]\right)_+ \tag{1.1.6}$$

for any sequence $b(n)$ since the random variables in question are discrete. For any $\vartheta>0$, it can be shown that $d_{b(n)} \xrightarrow{n\to\infty} 0$ if $b(n)=o(n)$. Hence, the joint distribution of non-macroscopic cycle counts converges in total variation distance to independent Poisson-distributed random variables. Since we consider convergence in total variation distance, it is possible to consider a growing (even diverging) number of cycle counts at the same time as long as $b(n)=o(n)$. For $\vartheta=1$, it has been proved by Barbour in **[6]** and independently in 1986 in some unpublished lecture notes by Diaconis and Pitman that $d_{b(n)} \le 2\frac{b(n)}{n}$ for all n. Arratia and Tavaré then showed in **[4]** in 1992 that, for $\vartheta=1$, $d_{b(n)} \le F(n/b(n))$ for a function F satisfying $\log(F(x)) \sim -x\log(x)$ as $x\to\infty$. So there is an exponential rate of convergence in this case, which seems to be specific to the uniform measure $\mathbb{P}_n^{(1)}$. The rate of convergence for all other known measures is at most algebraically fast. In particular, for the Ewens measure with cycle weights $1\neq\vartheta>0$, it was proved in **[3**, Theorems 3 and 5] that $d_{b(n)} = \mathcal{O}\left(\frac{b(n)}{n}\right)$ and that there is a lower bound of the form of $c\frac{b(n)}{n\log(n/b(n))}$ for some constant $c>0$ if $b(n)\to\infty$. When we prove an analogous result for random permutations without macroscopic cycles in Section 2.5, we show an upper bound for the rate of convergence which also decays at most algebraically fast.
We do not outline a proof here since we apply a similar argument in the course of Section 2.5. Note that analogous results have been derived for probability measures on other combinatorial structures (cf. again **[2]**).

1.1.2.3. *Functional Central Limit Theorem.* If we define

$$B_t(n) := \frac{\sum_{j=1}^{\lfloor n^t\rfloor} C_j - \vartheta t\log(n)}{\sqrt{\vartheta\log(n)}} \tag{1.1.7}$$

for $t\in[0,1]$, one can show that $(B_t(n))_{t\in[0,1]}$, under $\mathbb{P}_n^{(\vartheta)}$, converges in distribution (as a stochastic process) to the standard Brownian motion. This result was proved in **[18]** for $\vartheta=1$ and in **[34]** for general $\vartheta>0$. See also **[22]** and **[5]**. Observe that the central limit theorem in Equation (1.1.2) is a special case of the functional central limit theorem under consideration.
When interpreted with Equation (1.1.6) in mind, writing

$$B_t(n) = \frac{\sum_{j=1}^{\lfloor n^t\rfloor} Z_j - \vartheta t\log(n)}{\sqrt{\vartheta\log(n)}} + \frac{\sum_{j=1}^{\lfloor n^t\rfloor}(C_j-Z_j)}{\sqrt{\vartheta\log(n)}}$$

illustrates how a comparison with a process of independent component counts provides us with the idea of proof: To wit, the convergence of the process

$$\left(\frac{\sum_{j=1}^{\lfloor n^t\rfloor} Z_j - \vartheta t\log(n)}{\sqrt{\vartheta\log(n)}}\right)_{t\in[0,1]}$$

to the standard Brownian motion is easily proved. The fact that there is a coupling such that the error term

$$\left(\frac{\sum_{j=1}^{\lfloor n^t\rfloor}(C_j-Z_j)}{\sqrt{\vartheta\log(n)}}\right)_{t\in[0,1]}$$

converges to 0 in probability is a finding of a more refined analysis of the consequences of the Feller coupling (cf. **[3]**).
In Section 2.5.2, we will deduce that the functional central limit theorem presented here still applies to short cycles in the model of random permutations without macroscopic cycles.

1.1.3. Split-Merge Processes and Macroscopic Cycles. This section is based primarily on [**63, 28, 2**]. Let $\tau_{i_1 i_2}$ denote the transposition of two distinct elements $i_1, i_2 \in \{1, 2, \ldots, n\}$ and let σ be a permutation in S_n. If we consider the product $\tau_{i_1 i_2} \circ \sigma$, then the indices i_1 and i_2 belong to the same cycle in $\tau_{i_1 i_2} \circ \sigma$ if they belong to different cycles in σ. Conversely, if i_1 and i_2 are contained in the same cycle in σ, they will belong to different cycles in $\tau_{i_1 i_2} \circ \sigma$. Hence, multiplying the permutation σ with the transposition $\tau_{i_1 i_2}$ will either split the cycle which contains both i_1 and i_2 or it will merge the two distinct cycles to which the elements i_1 and i_2 belong. Recall that C is defined as the total number of cycles of a permutation. So we always have $C(\tau_{i_1 i_2} \circ \sigma) = C(\sigma) \pm 1$. We can now define a discrete-time Markov chain on the state space S_n which we call a split-merge process: Let $g_s, g_m \in [0, 1]$. Then

$$P(\sigma; \tau_{i_1 i_2} \circ \sigma) := \frac{1}{\frac{1}{2} n (n-1)} \begin{cases} g_s & \text{if } C(\tau_{i_1 i_2} \circ \sigma) = C(\sigma) + 1 \\ g_m & \text{if } C(\tau_{i_1 i_2} \circ \sigma) = C(\sigma) - 1 \end{cases} \tag{1.1.8}$$

and $P(\sigma; \sigma) := 1 - \sum_{i_1 < i_2} P(\sigma; \tau_{i_1 i_2} \circ \sigma)$ define a transition matrix P. Intuitively, first a transposition $\tau_{i_1 i_2}$ is sampled uniformly from the set of transpositions. Then, if multiplying σ with said transposition $\tau_{i_1 i_2}$ would split a cycle, this cycle is indeed split only with probability g_s. If multiplication of σ with $\tau_{i_1 i_2}$ were to merge two cycles, they are merged with probability g_m.
If π_t is the distribution of the Markov chain at time t, the distribution at time $t+1$ is given by

$$\pi_{t+1}(\sigma) = \pi_t(\sigma) P(\sigma; \sigma) + \sum_{i_1 < i_2} \pi_t(\tau_{i_1 i_2} \circ \sigma) P(\tau_{i_1 i_2} \circ \sigma; \sigma)$$

for any $\sigma \in S_n$ since $\sigma = \tau_{i_1 i_2} \circ (\tau_{i_1 i_2} \circ \sigma)$. Thus, the detailed balance condition translates into

$$\pi(\sigma) P(\sigma; \tau_{i_1 i_2} \circ \sigma) = \pi(\tau_{i_1 i_2} \circ \sigma) P(\tau_{i_1 i_2} \circ \sigma; \sigma).$$

Observe that the Ewens measure $\mathbb{P}_n^{(\vartheta)}$ fulfils the detailed balance condition if

$$\vartheta = \frac{g_s}{g_m},$$

so in this case the split-merge process is reversible and $\mathbb{P}_n^{(\vartheta)}$ is an invariant distribution. Furthermore, it is irreducible and also aperiodic for $0 < g_s, g_m < 1$. The general theory of Markov chains thus implies convergence of the split-merge process to equilibrium with an exponential rate (cf., e.g., [**42**]). A derived Markovian dynamic on the set $S_{n,\alpha}$ of permutations without macroscopic cycles will allow us to perform simulations of the measure $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ by applying Markov chain Monte Carlo (see Section A.2).
Moreover, if we project the Markov chain defined above onto integer partitions (i.e., we only keep track of the lengths of the cycles in non-increasing order), we obtain a split-merge process which is still Markovian. Dividing the lengths of the cycles by n and letting $n \to \infty$, we arrive at a split-merge process on the partitions of the unit interval $[0, 1]$. This approach is one way to conclude that the rescaled cycle lengths of ϑ-biased random permutations in non-increasing order converge to the so-called Poisson-Dirichlet distribution with parameter ϑ ($\mathrm{PD}(\vartheta)$) since it is an invariant distribution of the split-merge process on $[0, 1]$ (cf., e.g., [**65, 37, 64, 52, 20**] and the textbooks [**7, 28**]). This result provides information about macroscopic cycles since it rescales the cycle lengths by the factor of $1/n$. Put differently, the law of the fraction of indices contained in the longest, second longest etc. cycles converges weakly to the Poisson-Dirichlet distribution.
One of many ways to define $\mathrm{PD}(\vartheta)$ is the following: Let $(U_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. $\mathrm{Beta}(\vartheta)$-distributed random variables and $V_1 := U_1$, $V_2 := (1 - U_1) U_2$, $V_3 := (1 - U_1)(1 - U_2) U_3$, etc. Then the joint law of $(V_k)_{k \in \mathbb{N}}$ is called the $\mathrm{GEM}(\vartheta)$ distribution named for Griffiths, Engen, and McCloskey. Intuitively, we first break the unit interval into two parts of lengths $V_1 = U_1$ and $1 - U_1$, respectively. We keep the first one and again break the second one into two parts of lengths $V_2 = (1 - U_1) U_2$ and $(1 - U_1)(1 - U_2)$. Then we iterate this procedure, which is referred to as the stick-breaking process. The Poisson-Dirichlet distribution is obtained as the law of the $(V_k)_{k \in \mathbb{N}}$ in non-increasing order.
Observe that the connections between the Ewens measure and the Poisson-Dirichlet distribution may also be established by applying methods adapted to the framework of logarithmic combinatorial structures (cf. [**2**, Section 5.5]).

1.2. Weak Convergence and Moment-Generating Functions

Weak convergence is an essential concept of modern probability theory. This section intends to give a concise overview of the topic with the applications to follow in mind. Hence, special emphasis will be put on the connection between weak convergence on the one hand and convergence of moment-generating functions on the other hand. The results presented in this regard in Section 1.2.2 will be important points of reference throughout the thesis since moment-generating functions are objects naturally available to approaches grounded in analytic combinatorics (see Section 1.3). The theory of moment-generating functions thus enables us to apply analytic combinatorics to probabilistic models.

1.2.1. Weak Convergence. Let $(S, \mathcal{B}(S))$ be a metric space S endowed with its Borel σ-algebra $\mathcal{B}(S)$. If $\mathbb{P}, \mathbb{P}_1, \mathbb{P}_2, \ldots$ are probability measures on $(S, \mathcal{B}(S))$, we say that $\mathbb{P}_n$ converges weakly to $\mathbb{P}$, $\mathbb{P}_n \xrightarrow{w} \mathbb{P}$, when

$$\int_S f \mathrm{d}\mathbb{P}_n \xrightarrow{n\to\infty} \int_S f \mathrm{d}\mathbb{P}$$

holds for all bounded and continuous functions $f : S \to \mathbb{R}$. The following Portemanteau Theorem provides a number of useful equivalent characterizations of weak convergence.

PROPOSITION 1.2.1 (Portemanteau Theorem, [**36**, 4.25]). *For probability measures $\mathbb{P}, \mathbb{P}_1, \mathbb{P}_2, \ldots$ on a metric space $(S, \mathcal{B}(S))$, the following conditions are equivalent:*

(1) $\mathbb{P}_n \xrightarrow{w} \mathbb{P}$,
(2) $\liminf_{n\to\infty} \mathbb{P}_n[G] \geq \mathbb{P}[G]$ *for any open set* $G \subset S$,
(3) $\limsup_{n\to\infty} \mathbb{P}_n[F] \leq \mathbb{P}[F]$ *for any closed set* $F \subset S$,
(4) $\lim_{n\to\infty} \mathbb{P}_n[B] = \mathbb{P}[B]$ *for any* $B \in \mathcal{B}(S)$ *with* $\mathbb{P}[\partial B] = 0$.

A set $B \in \mathcal{B}(S)$ with $\mathbb{P}[\partial B] = 0$ is called a $\mathbb{P}$-continuity set.

It is possible to shift the perspective and consider random variables instead of probability measures: When $\boldsymbol{X}, \boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ are measurable maps from some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into $\mathbb{R}^K$ for $K \in \mathbb{N}$, we say that $\boldsymbol{X}_n$ converges in distribution to $\boldsymbol{X}$, $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$, if the distributions of $\boldsymbol{X}_n$, $\mathbb{P} \circ \boldsymbol{X}_n^{-1}$, converge weakly (as probability measures on $\mathbb{R}^K$) to the distribution of $\boldsymbol{X}$ which is given by $\mathbb{P} \circ \boldsymbol{X}^{-1}$. Let the cumulative distribution function $F_{\boldsymbol{X}}$ of $\boldsymbol{X} = (X_1, X_2, \ldots, X_K)$ be given by

$$F(\boldsymbol{x}) := \mathbb{P}[X_k \leq x_k \text{ for all } k \in \{1, 2, \ldots, K\}]$$

for $\boldsymbol{x} = (x_k)_{k=1}^K \in \mathbb{R}^K$ and define the characteristic function (CF) of $\boldsymbol{X}$ as

$$\chi_{\boldsymbol{X}}(\boldsymbol{s}) := \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\boldsymbol{s}\cdot\boldsymbol{X}}\right]$$

for $\boldsymbol{s} = (s_k)_{k=1}^K \in \mathbb{R}^K$, where $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$. We further have

PROPOSITION 1.2.2 ([**13**, Sections 25 and 29], [**36**, Theorem 5.3]). *Let $\boldsymbol{X}, \boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^K$. Then the following are equivalent:*

(1) $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$,
(2) $F_{\boldsymbol{X}_n}(\boldsymbol{x}) \to F_{\boldsymbol{X}}(\boldsymbol{x})$ *for any continuity point* $\boldsymbol{x}$ *of* $F_{\boldsymbol{X}}$,
(3) $\chi_{\boldsymbol{X}_n}(\boldsymbol{s}) \to \chi_{\boldsymbol{X}}(\boldsymbol{s})$ *for any* $\boldsymbol{s} \in \mathbb{R}^K$.

The classical result connecting characteristic functions and weak convergence is Lévy's Continuity Theorem. We call a function $f : \mathbb{R}^K \to \mathbb{C}$ partially continuous in $\boldsymbol{x} = (x_k)_{k=1}^K \in \mathbb{R}^K$ if $y_k \mapsto f(x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_K)$ is continuous for all k.

PROPOSITION 1.2.3 (Lévy's Continuity Theorem, [**43, 38**]). *Let $\boldsymbol{X}, \boldsymbol{X}_n, n \in \mathbb{N}$, be $\mathbb{R}^K$-valued random variables with characteristic functions $\chi_{\boldsymbol{X}}, \chi_{\boldsymbol{X}_n}$. Then the following hold:*

(1) If $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$, then $\chi_{\boldsymbol{X}_n}(\boldsymbol{s}) \to \chi_{\boldsymbol{X}}(\boldsymbol{s})$ for any $\boldsymbol{s} \in \mathbb{R}^K$ and uniformly on compact sets.
(2) If there is a function $\chi : \mathbb{R}^K \to \mathbb{C}$ which is partially continuous in 0 such that $\chi_{\boldsymbol{X}_n}(\boldsymbol{s}) \to \chi(\boldsymbol{s})$ for any $\boldsymbol{s} \in \mathbb{R}^K$, then there exists a random variable $\boldsymbol{Y}$ such that $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{Y}$ and $\chi_{\boldsymbol{Y}} = \chi$.

Proposition 1.2.3 will serve as a backdrop for the presentation of the connection between moment-generating functions and weak convergence.

1.2.2. Moment-Generating Functions. The moment-generating function or MGF of an $\mathbb{R}^K$-valued random variable $\boldsymbol{X}$ is defined by

$$M_{\boldsymbol{X}}(\boldsymbol{s}) = \mathbb{E}\left[\mathrm{e}^{\boldsymbol{s}\cdot\boldsymbol{X}}\right]$$

for $\boldsymbol{s} \in \mathbb{R}^K$ such that $\mathrm{e}^{\boldsymbol{s}\cdot\boldsymbol{X}}$ is integrable. We aim for an analogue of Proposition 1.2.3 which features moment-generating instead of characteristic functions. The main reason for applying the theory of MGFs instead of CFs in this thesis is that they can be derived much more easily by means of analytic combinatorics since they give rise to generating functions related to combinatorial structures which have non-negative coefficients (see Section 1.3 and, in particular, Section 1.3.5). Put in the terms of Section 2.1 below, we require the weights $q_{j,n}$ to be non-negative. Moreover, the question of integrability of $\mathrm{e}^{\boldsymbol{s}\cdot\boldsymbol{X}}$ will not be an issue for the applications in question.
Observe further that, if we wanted to prove convergence in distribution with the help of characteristic functions, we would have to establish pointwise convergence of the respective CFs for all $\boldsymbol{s}$ in order to apply Proposition 1.2.3. Pointwise convergence in some interval in the one-dimensional case, for instance, would not be sufficient since two different characteristic functions may coincide on some interval (cf. [**48**]). If a moment-generating function exists in some interval containing the origin, however, it uniquely determines the distribution of the underlying random variable (see, e.g., [**17**]). As we will see in Proposition 1.2.5, in the case of MGFs we only have to consider arguments $\boldsymbol{s}$ in some set with non-empty interior. In the applications below, this set will be given by $\left\{\boldsymbol{s} \in \mathbb{R}^K : s_k \geq 0 \text{ for all } k\right\}$. The restriction to this set will simplify proofs considerably as a comparison between, e.g., the proofs of Propositions 2.4.1 and 2.4.4 shows. In the terms of Section 2.1, the assumption $s_k \geq 0$ for all k eases verifying admissibility.
Since some of the results have only been established recently, we give a short account of the development following [**48, 69**]. The classical result from 1942 is

PROPOSITION 1.2.4 (Curtiss' Theorem, [**17**]). *Let $s_1 > 0$ and let M_{X_n} be the moment-generating functions of real-valued random variables X_n such that $M_{X_n}(s)$ exists for all $|s| < s_1$. Assume further that there is a real function M and $0 < s_2 < s_1$ such that $M_n(s) \xrightarrow{n\to\infty} M(s)$ for all $|s| \leq s_2$. Then there is a real-valued random variable X such that $X_n \xrightarrow{d} X$ and $M(s) = M_X(s)$ for all $|s| \leq s_2$ hold.*

Other than in Lévy's Continuity Theorem, however, a converse statement which concludes convergence of the MGFs from convergence in distribution does not hold in general (see [**17**] for an example). The proof applies Vitali's Theorem (cf., e.g., [**56**, Section 2.4]) and Lévy's Continuity Theorem. See also [**55**, Section 1.2.3].
In 2006, it was shown that the interval in which the MGFs converge need not contain the origin.

PROPOSITION 1.2.5 ([**48**, Theorem 2]). *Let $0 < a < b$ and let $X, X_n, n \in \mathbb{N}$, be real-valued random variables such that the moment-generating functions $M_X(s), M_{X_n}(s), n \in \mathbb{N}$, are finite for $s \in (a, b)$. If $M_{X_n}(s) \to M_X(s)$ for any $s \in (a, b)$, then $X_n \xrightarrow{d} X$.*

The proof of Proposition 1.2.5 considers the cumulative distribution functions F_{X_n} and establishes that condition (2) in Proposition 1.2.2 holds. It does so by first defining a sequence of suitable auxiliary cumulative distribution functions G_{Y_n} related to F_{X_n} which converge due to Curtiss' Theorem. The remaining part of the proof is concerned with deriving convergence of F_{X_n} from convergence of G_{Y_n}.
The following generalization of the previous results to the multi-dimensional case due to Yakymiv in 2011 considers σ-finite measures instead of only probability measures. This is why the measures are only defined on a δ-ring and the concept of convergence is slightly adapted. In Corollary 1.2.7, we will restate parts of the content of Proposition 1.2.6 in a form which is adjusted to the applications in this thesis.
Recall that a δ-ring $\mathcal{R}$ is a non-empty collection of sets such that

(1) $B_1 \cup B_2 \in \mathcal{R}$ if $B_1, B_2 \in \mathcal{R}$,
(2) $B_1 \setminus B_2 \in \mathcal{R}$ if $B_1, B_2 \in \mathcal{R}$, and
(3) $\bigcap_{n\in\mathbb{N}} B_n \in \mathcal{R}$ if $B_n \in \mathcal{R}$ for all $n \in \mathbb{N}$

hold. Let $\mathcal{R}_K$ be the δ-ring which consists of the bounded Borel sets of $\mathbb{R}^K$. We say that a sequence of σ-finite measures U_n defined on $\mathcal{R}_K$ converges Y-weakly to the σ-finite measure U on $\mathcal{R}_K$ if $U_n[B] \xrightarrow{n\to\infty} U[B]$ for any $B \in \mathcal{R}_K$ such that $U[\partial B] = 0$.

PROPOSITION 1.2.6 ([**69**, Theorem 2]). *Suppose that for a sequence of σ-finite measures U_n, $n \in \mathbb{N}$, defined on $\mathcal{R}_K$ the moment-generating functions*

$$M_{U_n}(\boldsymbol{s}) = \int_{\mathbb{R}^K} \mathrm{e}^{\boldsymbol{s}\cdot\boldsymbol{v}} U_n(\mathrm{d}\boldsymbol{v})$$

exist for all $\boldsymbol{s}$ in a domain D with non-empty interior. Then the following assertions hold:

(1) If

$$M_{U_n}(\boldsymbol{s}) \xrightarrow{n\to\infty} M(\boldsymbol{s}) < \infty \tag{1.2.1}$$

for some function M and any $\boldsymbol{s} \in D$, then M is the moment-generating function of a σ-finite measure U on $\mathcal{R}_K$ and U_n converges Y-weakly to U as $n \to \infty$.

(2) Let D be bounded. If U_n converges Y-weakly to U and if the $M_{U_n}(\boldsymbol{s}_0)$ are bounded for an arbitrary $\boldsymbol{s}_0 \in \partial D$, then Equation (1.2.1) holds, the moment-generating function $M_U(\boldsymbol{s})$ exists for any $\boldsymbol{s} \in D$ and $M_U(\boldsymbol{s}) = M(\boldsymbol{s})$ for any $\boldsymbol{s} \in D$.

The proof involves two parts. In the first part it establishes that moment-generating functions which exist on a domain D separate measures. It does so by considering analytic continuations of two moment-generating functions M_{U_1} and M_{U_2} which coincide on D. By the uniqueness theorem for analytic functions, the resulting analytic functions are still the same. They are then interpreted as related to characteristic functions of two auxiliary probability measures $\tilde{U}_1$ and $\tilde{U}_2$ which also have to coincide since characteristic functions separate probability measures. An application of the Radon-Nikodym theorem then derives $U_1 = U_2$ from $\tilde{U}_1 = \tilde{U}_2$.
The second part of the proof deals with the statements proper of Proposition 1.2.6. It starts with the second item and divides the integration domain in the definition of the MGFs into a suitable compact U-continuity set B_m (with $B_m \uparrow \mathbb{R}^K$ as $m \to \infty$) and its complement. By Y-weak convergence, $\int_{B_m} \mathrm{e}^{\boldsymbol{s}\cdot\boldsymbol{v}} U_n(\mathrm{d}\boldsymbol{v}) \xrightarrow{n\to\infty} \int_{B_m} \mathrm{e}^{\boldsymbol{s}\cdot\boldsymbol{v}} U(\mathrm{d}\boldsymbol{v})$ for all m. The assumption concerning $\boldsymbol{s}_0 \in \partial D$ can be applied to conclude $\sup_{n\in\mathbb{N}} \int_{B_m^C} \mathrm{e}^{\boldsymbol{s}\cdot\boldsymbol{v}} U_n(\mathrm{d}\boldsymbol{v}) \xrightarrow{m\to\infty} 0$, and $\int_{B_m^C} \mathrm{e}^{\boldsymbol{s}\cdot\boldsymbol{v}} U(\mathrm{d}\boldsymbol{v}) \xrightarrow{m\to\infty} 0$ also holds. So item (2) follows from the triangle inequality.
The proof of item (1) derives tightness of the measures from $\sup_{n\in\mathbb{N}} U_n[B] < \infty$ for all bounded Borel sets B, which is a consequence of $M_{U_n}(\boldsymbol{s}) \xrightarrow{n\to\infty} M_U(\boldsymbol{s})$ and the inequality $M_{U_n}(\boldsymbol{s}) \geq U_n[B] \inf_{\boldsymbol{v}\in B} \mathrm{e}^{\boldsymbol{s}\cdot\boldsymbol{v}}$ for $\boldsymbol{s} \in D$ and then applies item (2).
We now give the statement which will be applied numerous times in the following sections.

COROLLARY 1.2.7. *Let $\boldsymbol{X}$ and $\boldsymbol{X}_n, n \in \mathbb{N}$, be $\mathbb{R}^K$-valued random variables with moment-generating functions $M_{\boldsymbol{X}}$ and $M_{\boldsymbol{X}_n}, n \in \mathbb{N}$, such that $M_{\boldsymbol{X}_n}(\boldsymbol{s}) \xrightarrow{n\to\infty} M_{\boldsymbol{X}}(\boldsymbol{s})$ for any $\boldsymbol{s}$ in a domain $D \in \mathcal{B}(\mathbb{R}^K)$ with non-empty interior. If either*

(1) the distribution of $\boldsymbol{X}$ has a density with respect to Lebesgue measure,
(2) $\boldsymbol{X}$ and $\boldsymbol{X}_n, n \in \mathbb{N}$, almost surely adopt only values in the same discrete set $B_0 \subset \mathbb{R}^K$ or
(3) $\boldsymbol{X}$ is almost surely constant,

we have $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$ as $n \to \infty$.

PROOF. By Proposition 1.2.6, we know that the distributions of $\boldsymbol{X}_n$ converge Y-weakly to the distribution of $\boldsymbol{X}$, i.e. $\mathbb{P}[\boldsymbol{X}_n \in B] \to \mathbb{P}[\boldsymbol{X} \in B]$ for any $B \in \mathcal{R}_K$ with $\mathbb{P}[\boldsymbol{X} \in \partial B] = 0$. We have to show the same statement, but for any $B \in \mathcal{B}(\mathbb{R}^K)$ such that $\mathbb{P}[\boldsymbol{X} \in \partial B] = 0$.
Assume (1) and let $\epsilon > 0$. By continuity from below, there is $m_0 \in \mathbb{N}$ such that $\mathbb{P}[\boldsymbol{X} \notin B_{m_0}(\boldsymbol{0})] < \frac{\epsilon}{4}$, where $B_{m_0}(\boldsymbol{0})$ denotes the ball of radius m_0 centered at $\boldsymbol{0}$. Because $B \cap B_{m_0}(\boldsymbol{0}) \in \mathcal{R}_K$ and $\mathbb{P}[\boldsymbol{X} \in \partial(B \cap B_{m_0}(\boldsymbol{0}))] = 0$ (this is a consequence of $\partial(B \cap B_{m_0}(\boldsymbol{0})) \subset \partial B \cup \partial B_{m_0}(\boldsymbol{0})$, $\mathbb{P}[\boldsymbol{X} \in \partial B] = 0$ and the existence of a density) as well as $B_{m_0}(\boldsymbol{0}) \in \mathcal{R}_K$ and $\mathbb{P}[\boldsymbol{X} \in \partial B_{m_0}(\boldsymbol{0})] = 0$ hold, there is $n_0 \in \mathbb{N}$ such that

$$|\mathbb{P}[\boldsymbol{X} \in B \cap B_{m_0}(\boldsymbol{0})] - \mathbb{P}[\boldsymbol{X}_n \in B \cap B_{m_0}(\boldsymbol{0})]| < \frac{\epsilon}{4}$$

and

$$|\mathbb{P}[\boldsymbol{X} \in B_{m_0}(\boldsymbol{0})] - \mathbb{P}[\boldsymbol{X}_n \in B_{m_0}(\boldsymbol{0})]| < \frac{\epsilon}{4}$$

for all $n \geq n_0$. Altogether we obtain

$$
\begin{aligned}
&\left|\mathbb{P}\left[\boldsymbol{X} \in B\right]-\mathbb{P}\left[\boldsymbol{X}_n \in B\right]\right| \\
\leq &\left|\mathbb{P}\left[\boldsymbol{X} \in B \cap B_{m_0}(\mathbf{0})\right]-\mathbb{P}\left[\boldsymbol{X}_n \in B \cap B_{m_0}(\mathbf{0})\right]\right|+\mathbb{P}\left[\boldsymbol{X} \in B_{m_0}^C(\mathbf{0})\right]+\mathbb{P}\left[\boldsymbol{X}_n \in B_{m_0}^C(\mathbf{0})\right] \\
\leq &\frac{\epsilon}{4}+\frac{\epsilon}{4}+1-\mathbb{P}\left[\boldsymbol{X}_n \in B_{m_0}(\mathbf{0})\right].
\end{aligned}
$$

Due to

$$
\begin{aligned}
\mathbb{P}\left[\boldsymbol{X}_n \in B_{m_0}(\mathbf{0})\right] \geq &\mathbb{P}\left[\boldsymbol{X} \in B_{m_0}(\mathbf{0})\right]-\frac{\epsilon}{4} \\
\geq &1-\frac{1}{2}\epsilon,
\end{aligned}
$$

we have

$$
\left|\mathbb{P}\left[\boldsymbol{X} \in B\right]-\mathbb{P}\left[\boldsymbol{X}_n \in B\right]\right| \leq \epsilon.
$$

So $\boldsymbol{X}_n$ converges in distribution to $\boldsymbol{X}$.
In the case of (2), we only have to verify that $\mathbb{P}\left[\boldsymbol{X}_n=\boldsymbol{v}_0\right] \xrightarrow{n \rightarrow \infty} \mathbb{P}\left[\boldsymbol{X}=\boldsymbol{v}_0\right]$ holds for any $\boldsymbol{v}_0 \in B_0$. This statement follows from Y-weak convergence and B_0 being discrete. Case (3) is an immediate consequence of Y-weak convergence. $\square$

1.2.3. Weak Convergence on Function Spaces. For the convenience of the reader, this section assembles tools for establishing weak convergence of probability measures on certain function spaces which will be applied in the course of this thesis.
Define the Skorohod space $\mathcal{D}[0,1]$ as the space of real-valued càd-làg functions on $[0,1]$ (i.e., they are right-continuous and have left limits) and endow it with the Borel σ-algebra with respect to the Skorohod topology. Let Λ be the set of strictly increasing, bijective, and continuous maps $\lambda:[0,1] \rightarrow[0,1]$ and $\|\cdot\|_{\infty}$ the supremum norm. Then the Skorohod topology is, for instance, generated by the metric

$$
d_1(X, Y)=\inf_{\lambda \in \Lambda}\left(\|\lambda-\mathrm{id}\|_{\infty} \vee\|X-Y \circ \lambda\|_{\infty}\right) \tag{1.2.2}
$$

for $X, Y \in \mathcal{D}[0,1]$, where $\mathrm{id}:[0,1] \rightarrow[0,1]$ is the identity map. On an intuitive level, according to the definition, X and Y are close if small deformations in both "space" and "time" may transform Y into X. The additional freedom of deforming "time" when compared to the usual supremum norm on the space of continuous functions $\mathcal{C}[0,1]$ is essential for dealing with jump processes.
Since the Skorohod space is metrizable, we are in the general situation of Section 1.2.1 and thus possess a concept of weak convergence of probability measures on $\mathcal{D}[0,1]$. The general strategy of proving weak convergence in this setting consists in showing tightness and convergence of the finite-dimensional distributions. An important criterion in the context of this thesis is given in

PROPOSITION 1.2.8 ([**14**, Theorem 13.5 and Equation (13.14)]). *Let $\left(\mathbb{P}_n\right)_{n \in \mathbb{N}}$ and $\mathbb{P}$ be probability measures on $\mathcal{D}[0,1]$ and let $X_t: \mathcal{D}[0,1] \rightarrow \mathbb{R}$ for $t \in[0,1]$ be the value of the function $X \in \mathcal{D}[0,1]$ at t. Suppose that*

$$
\mathbb{P}_n \circ\left(X_{t_1}, X_{t_2}, \ldots, X_{t_K}\right)^{-1} \xrightarrow{w} \mathbb{P} \circ\left(X_{t_1}, X_{t_2}, \ldots, X_{t_K}\right)^{-1}
$$

as $n \rightarrow \infty$ for all points $t_1, \ldots, t_K \in[0,1]$ such that X is $\mathbb{P}$-almost surely continuous in t_k, that

$$
\mathbb{P} \circ X_{1-\delta}^{-1} \xrightarrow{w} \mathbb{P} \circ X_1^{-1}
$$

as $\delta \rightarrow 0$ and that there is $\alpha>\frac{1}{2}, \beta \geq 0$, and a non-decreasing continuous function F on $[0,1]$ such that

$$
\mathbb{E}_n\left[\left|X_t-X_{t_1}\right|^{2 \beta}\left|X_{t_2}-X_t\right|^{2 \beta}\right] \leq\left|F\left(t_2\right)-F\left(t_1\right)\right|^{2 \alpha}
$$

for all $0 \leq t_1 \leq t \leq t_2 \leq 1$. Then,

$$
\mathbb{P}_n \xrightarrow{w} \mathbb{P}.
$$

The statements about and the definition of $\mathcal{D}[0,1]$ generalize in a natural way to $\mathcal{D}[0, T]$ for $T>0$. One can also consider the space $\mathcal{D}[0, \infty)$ of càd-làg functions defined on the real half-axis endowed with the Skorohod topology and its Borel σ-algebra. Let $\left.X\right|_{[0, T]}$ denote the restriction of $X \in \mathcal{D}[0,\infty)$ to $[0,T]$. Then we have $X|_{[0,T]} \in \mathcal{D}[0,T]$ and one can show that the restriction is measurable. The corresponding metric on $\mathcal{D}[0,\infty)$ is given by

$$d_\infty(X,Y) = \sum_{m=1}^{\infty} 2^{-m} \left(d_m \left(X|_{[0,m]}, Y|_{[0,m]} \right) \wedge 1 \right) \tag{1.2.3}$$

for $X, Y \in \mathcal{D}[0,\infty)$. In this case, weak convergence of probability measures on $\mathcal{D}[0,\infty)$ can be traced back to weak convergence on $\mathcal{D}[0,T]$.

PROPOSITION 1.2.9 ([**14**, Theorem 16.7]). *Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$ and $\mathbb{P}$ be probability measures on $\mathcal{D}[0,\infty)$. Then we have*

$$\mathbb{P}_n \xrightarrow{w} \mathbb{P}$$

if and only if

$$\mathbb{P}_n \circ X|_{[0,T]}^{-1} \xrightarrow{w} \mathbb{P}_n \circ X|_{[0,T]}^{-1}$$

for every $T > 0$ such that X is $\mathbb{P}$-almost surely continuous in T.

1.3. Generating Functions and Analytic Combinatorics

Analytic combinatorics is the mathematical discipline which investigates combinatorial structures with analytic methods. Typical questions which arise in combinatorics such as "How many permutations of size n are there for large n?" are transformed into questions about the behaviour of certain analytic functions which encode the relevant information. Such functions are called generating functions and will be presented in Section 1.3.1. If a combinatorial structure is given, following this approach typically necessitates two steps: First, one has to find the generating function in question. The central technique for solving this problem for a wide range of structures built from smaller blocks is the symbolic method discussed in Section 1.3.2. Section 1.3.3 offers a shortcut for permutations which is a special case of Pólya's Enumeration Theorem. Understanding its proof does not require the content of Section 1.3.2, but the symbolic method provides useful motivation and background as to why this specific generating function arises in the given context. If the pertinent generating function has been obtained, then one still has to extract the desired information. This is the second step and may be referred to as "counting by integrating". Three techniques relevant for many applications, including singularity analysis, are presented in Section 1.3.4. Yet the instrument of choice in a variety of cases, which we will employ in the study of random permutations without marcoscopic cycles, is the saddle-point method. It will therefore be introduced in more detail in Section 1.3.5 and eventually applied in Section 2.1.
The veritable summa of the field and main source for this section is the book [**30**].

1.3.1. Generating Functions.

There are different kinds of generating functions of importance for analytic combinatorics. Ordinary generating functions are adapted to counting so-called unlabelled objects (e.g., words and trees), whereas exponential generating functions perform the same role for labelled objects (e.g., permutations). Labelled and unlabelled objects will be precisely defined in Section 1.3.2. If one wants to obtain more information than just the number of certain objects of a given size, more parameters have to be introduced. One therefore arrives at the concept of bivariate or, more generally, multivariate generating functions. Further important applications of generating functions include the analysis of algorithms and recurrence relations (cf. [**57**]).

1.3.1.1. *Ordinary Generating Functions.* We start by giving the definition.

DEFINITION 1.3.1. The ordinary generating function (OGF) of a real-valued sequence $(A_n)_{n\in\mathbb{N}_0}$ is given by the formal power series

$$A(z) := \sum_{n=0}^{\infty} A_n z^n.$$

Depending on the concrete sequence $(A_n)_{n\in\mathbb{N}_0}$, one may determine a radius of convergence and consider $A(z)$ as an analytic function on a suitable complex domain. The next definition provides a useful notation for extracting the coefficients of a formal power series.

DEFINITION 1.3.2. Let $A(z) = \sum_{n=0}^{\infty} A_n z^n$ be a formal power series. Then coefficient extraction is defined by

$$[z^n]\, A(z) := A_n$$

for $n \in \mathbb{N}_0$.

Some elementary examples (see [**57**, Table 3.1]) of ordinary generating functions are

- $A(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $A_n = 1$,
- $A(z) = \frac{z}{(1-z)^2}$ for $A_n = n$,
- $A(z) = \log\left(\frac{1}{1-z}\right) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ for $A_0 = 0$, $A_n = \frac{1}{n}$ $(n \geq 1)$, and
- $A(z) = \frac{1}{1-z}\log\left(\frac{1}{1-z}\right) = \sum_{n=1}^{\infty} H_n z^n$ for $A_0 = 0$, $A_n = H_n$ $(n \geq 1)$. Here, $H_n = \sum_{j=1}^{n} \frac{1}{j}$ is the n-th harmonic number.

The particular strength of the concept of generating function consists in establishing relations between operations on sequences $(A_n)_{n\in\mathbb{N}_0}$ and operations on their respective generating functions $A(z)$. A small selection of such connections is given in

Proposition 1.3.3 ([**57**, Theorem 3.1]). *Let* $(A_n)_{n\in\mathbb{N}_0}$ *and* $(B_n)_{n\in\mathbb{N}_0}$ *be real-valued sequences with ordinary generating functions* $A(z)$ *and* $B(z)$*, respectively, which are analytic on a common complex domain. Then,*

- $A(z)+B(z)$ *is the OGF for* $A_0+B_0,\ A_1+B_1,\ A_2+B_2,\ldots,$
- $A'(z)$ *is the OGF for* $A_1,\ 2A_2,\ 3A_3,\ldots,$
- $zA(z)$ *is the OGF for* $0,\ A_0,\ A_1,\ldots,$ *and*
- $A(z)B(z)$ *is the OGF for* $A_0B_0,\ A_0B_1+A_1B_0,\ A_0B_2+A_1B_1+A_0B_2,\ldots$

The statements in Proposition 1.3.3 are easily proved.

1.3.1.2. *Exponential Generating Functions.* Exponential and ordinary generating function of a given sequence differ in factors of $\frac{1}{n!}$ in each summand.

Definition 1.3.4. The exponential generating function (EGF) of a real-valued sequence $(A_n)_{n\in\mathbb{N}_0}$ is given by the formal power series

$$A(z) := \sum_{n=0}^{\infty} \frac{A_n}{n!} z^n.$$

Again, as the case may be, $A(z)$ can be considered as an analytic function. Some elementary exponential generating functions are

- $A(z) = \mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $A_n = 1$,
- $A(z) = z\mathrm{e}^z = \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!}$ for $A_n = n$,
- $A(z) = \mathrm{e}^{cz} = \sum_{n=0}^{\infty} \frac{c^n z^n}{n!}$ for $c \in \mathbb{R}$ and $A_n = c^n$ as well as
- $A(z) = \frac{1}{2}z^2\mathrm{e}^z = \frac{1}{2}\sum_{n=2}^{\infty} \frac{z^n}{(n-2)!}$ for $A_0 = A_1 = 0$, $A_n = \binom{n}{2}$ $(n > 1)$ (see [**57**, Table 3.3]).

The differences between the definitions of ordinary and exponential generating functions entail small, but vital changes for the connection between sequences and generating functions.

Proposition 1.3.5 ([**57**, Theorem 3.2]). *Let* $(A_n)_{n\in\mathbb{N}_0}$ *and* $(B_n)_{n\in\mathbb{N}_0}$ *be real-valued sequences with exponential generating functions* $A(z)$ *and* $B(z)$*, respectively, which are analytic on a common complex domain. Then,*

- $A(z)+B(z)$ *is the EGF for* $A_0+B_0,\ A_1+B_1,\ A_2+B_2,\ldots,$
- $A'(z)$ *is the EGF for* $A_1,\ A_2,\ A_3,\ldots,$
- $zA(z)$ *is the EGF for* $0,\ A_0,\ 2A_1,\ 3A_2,\ldots,$ *and*
- $A(z)B(z)$ *is the EGF for* $A_0B_0,\ A_0B_1+A_1B_0,\ A_0B_2+2A_1B_1+A_0B_2,\ldots$

1.3.1.3. *Bivariate and Multivariate Generating Functions.* If we are interested in more than just the number of certain objects of a given size, we have to refine the concept of generating function in such a way that it is suitable for encoding the information in question. For instance, one might want to investigate the typical number of components of a combinatorial object. It is therefore necessary to introduce additional variables.

Definition 1.3.6. The bivariate generating function (BGF) of a doubly indexed, real-valued sequence $(A_{nl})_{(n,l)\in\mathbb{N}_0^2}$ is given by the formal power series

$$A(z,s) := \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} A_{nl} s^l \frac{z^n}{n!}.$$

In this case we enriched the structure of an EGF. Note that we might also define the bivariate generating function without the factor of $1/n!$, it would then extend the concept of OGF. Coefficient extraction is to be defined by

$$\left[z^n s^l\right] A(z,s) := \frac{A_{nl}}{n!}.$$

Bivariate generating functions are a useful tool and closely related to probability-generating functions: Assume that $A_{nl} \geq 0$ for all $n, l \in \mathbb{N}_0$ and that $\sum_{l=0}^{\infty} A_{nl} < \infty$ for all natural numbers n. If $P_{X_n}(s)$ denotes the probability-generating function of the random variable X_n which assumes the value $l \in \mathbb{N}_0$ with probability $\frac{A_{nl}}{\sum_{l\in\mathbb{N}_0} A_{nl}}$, then

$$P_{X_n}(s) = \frac{[z^n]\,A(z,s)}{[z^n]\,A(z,1)}.$$

The generalization from bivariate to multivariate generating functions is immediate and facilitates the study of $\mathbb{R}^d$-valued random variables.

1.3.2. The Symbolic Method. The focus of this section is the connection between combinatorical objects, i.e. objects which can be constructed according to certain rules in a finite way, and generating functions. The linchpin is the correspondence between a large variety of such rules on the one hand and operations on generating functions on the other hand.
We start by defining the generating function of a combinatorial class in an abstract way.

DEFINITION 1.3.7. A combinatorial class $\mathcal{A}$ is a finite or denumerable set endowed with a size function $|\cdot|_{\mathcal{A}} : \mathcal{A} \to \mathbb{N}_0$ such that the number of elements of any given size is finite. Let $\mathcal{A}_n$ be the set of elements in $\mathcal{A}$ of size n. Then the OGF (EGF) of $\mathcal{A}$ is the OGF (EGF) of its counting sequence $A_n = |\mathcal{A}_n|$.

In the following, two kinds of combinatorial objects (and classes) will have to be distinguished as their constructions differ from each other.

1.3.2.1. *Unlabelled Classes and Constructions.* An unlabelled class is just a combinatorial class according to Definition 1.3.7. The simplest construction is the usual Cartesian product defined in such a way that it respects the notion of size.

DEFINITION 1.3.8. The unlabelled Cartesian product of two combinatorial classes $\mathcal{B}$ and $\mathcal{C}$ is the set of ordered pairs

$$\mathcal{A} = \mathcal{B} \times \mathcal{C}$$

endowed with the size function $|\alpha|_{\mathcal{A}} = |\beta|_{\mathcal{B}} + |\gamma|_{\mathcal{C}}$ for all $\alpha = (\beta, \gamma) \in \mathcal{A}$.

One easily recognizes that the OGF of $\mathcal{A}$ is given by

$$A(z) = B(z) C(z)$$

since

$$A_n = \sum_{j=0}^{n} B_j C_{n-j}$$

for all $n \in \mathbb{N}_0$. Here, $B(z)$ and $C(z)$ are the ordinary generating functions of $\mathcal{B}$ and $\mathcal{C}$, respectively. For technical reasons it is convenient to introduce the neutral class $\mathcal{E}$ consisting of exactly one element ϵ of size 0.

DEFINITION 1.3.9. The sequence construction applied to a combinatorial class $\mathcal{A}$ is given by

$$\mathrm{SEQ}(\mathcal{A}) := \mathcal{E} \cup \mathcal{A} \cup \mathcal{A} \times \mathcal{A} \cup \ldots \tag{1.3.1}$$

Since the union in Equation (1.3.1) is disjoint, the corresponding OGF is

$$\sum_{n=0}^{\infty} A(z)^n = \frac{1}{1 - A(z)}.$$

An easy example may prove enlightening.

EXAMPLE 1.3.10. We consider binary words whose size is their length. Let $\mathcal{A} = \{0, 1\}$ be the alphabet and $\mathcal{W}_n = \mathcal{A}^n$ be the class of binary words of length n. Then the OGF of $\mathcal{A}$ is $A(z) = 2z$, and we obtain for the OGF of $\mathcal{W}_n$ that $W_n(z) = A(z)^n = 2^n z^n$. Clearly, the class of binary words $\mathcal{W} = \bigcup_{n \in \mathbb{N}} \mathcal{W}_n = \mathrm{SEQ}(\mathcal{A})$ then has the OGF $W(z) = \sum_{n=0}^{\infty} 2^n z^n = \frac{1}{1-2z}$.

The next examples illustrates the power of counting via generating functions.

EXAMPLE 1.3.11. Let $\mathcal{G}$ be the class of rooted plane general trees. A tree being plane here means that the ordering of subtrees matters. It is general when all degrees are admissible. The size of a tree is its number of vertices. Let $\mathcal{Z} = \{\cdot\}$ be the class which consists of exactly one tree of size 1. Since a subtree attached to the root is in itself a rooted plane general tree and there may be any finite number of such subtrees, we can establish a bijection between $\mathcal{G}$ and $\mathcal{Z} \times \mathrm{SEQ}(\mathcal{G})$. In the language of OGF we therefore obtain

$$G(z) = z \cdot \frac{1}{1 - G(z)},$$

which is equivalent to $G(z) - G(z)^2 - z = 0$. This equation is solvable by radicals. Because $G(z)$ has non-negative coefficients (they form a counting sequence), a calculation yields

$$G(z) = \frac{1}{2}\left(1 - \sqrt{1-4z}\right) = \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} z^n = \sum_{n=1}^{\infty} C_{n-1} z^n,$$

where C_n denotes the nth Catalan number.

There are far more unlabelled constructions which translate into natural operations on OGF (cf. [**30**, Theorem I.1]) and warrant presentation. Since permutations, the central topic of this thesis, are the prototype of labelled classes, however, our discussion will concentrate on labelled constructions in the next section.

1.3.2.2. *Labelled Classes and Constructions.* We start by defining labelled objects and classes.

DEFINITION 1.3.12. A labelled object of size $n \in \mathbb{N}$ is a graph consisting of n vertices such that its vertices bear distinct labels from $[n] := \{1, 2, \ldots, n\}$. A labelled class is a combinatorial class consisting of labelled objects.

The graphs referred to in Definition 1.3.12 may be chosen to have any typical graph property (e.g., they may be rooted or the ordering of subgraphs or connected components might be taken into consideration). In particular, they can be directed or undirected. Due to this leeway concerning the graphs under consideration, the definition is very broad in scope.

EXAMPLE 1.3.13. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma_1 & \sigma_2 & \ldots & \sigma_n \end{pmatrix}.$$

Since it can be encoded by the sequence $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, its representation as the directed graph

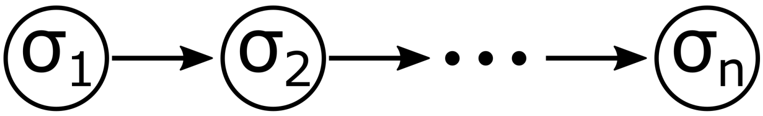

FIGURE 1.3.1. Representation of the permutation σ as a directed graph

shows that it is a labelled object. The EGF of the class $\mathcal{P}_n$ of permutations of n elements is $P_n(z) = z^n$.

The labels and their possible orderings are the reason why the exponential generating function with its factor of $1/n!$ is the tool of choice for describing labelled classes. The following definition of the analogue of the unlabelled Cartesian product construction (Definition 1.3.8) will elucidate the precise role played by the labels.

DEFINITION 1.3.14. The labelled product $\mathcal{A} = \mathcal{B} \star \mathcal{C}$ of two labelled classes $\mathcal{B}$, $\mathcal{C}$ is obtained by forming ordered pairs from $\mathcal{B} \times \mathcal{C}$ and performing all possible relabellings.

Clearly, since

$$A_n = \sum_{j=0}^{n} \binom{n}{j} B_j C_{n-j},$$

the EGF of $\mathcal{A}$ is given by

$$A(z) = B(z)\, C(z)$$

in this case. Note also that the labelled product is associative.

EXAMPLE 1.3.15. We want to determine the labelled product $\mathcal{P}_n \star \mathcal{P}_l$. An ordered pair from $\mathcal{P}_n \times \mathcal{P}_l$ is a pair of directed graphs of the form

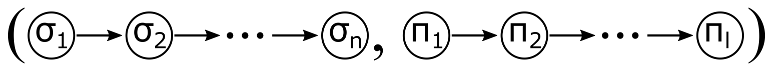

FIGURE 1.3.2. An ordered pair of two permutations σ and π represented as directed graphs

where $\sigma_j \in [n]$ and $\pi_j \in [l]$. Possible relabellings then lead to pairs of labelled graphs

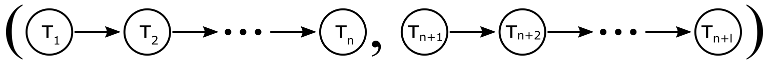

FIGURE 1.3.3. Relabelled pair of permutations

with $\tau_j \in [n+l]$. An immediate bijection (we drop the parentheses and connect the nodes with labels τ_n and τ_{n+1}) therefore shows that we can identify $\mathcal{P}_n \star \mathcal{P}_l$ and $\mathcal{P}_{n+l}$. This is consistent with $P_n(z)\, P_l(z) = z^{n+l} = P_{n+l}(z)$. The labelled class

$$\mathcal{P} = \mathcal{E} \cup \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$$

of permutations of any size then has the EGF

$$P(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

since it arises from a disjoint union. Here, consider the neutral class $\mathcal{E}$, which consists of exactly one element of size 0, also as an instance of a labelled class.

So far, we only have recovered the well-known fact that there are $n!$ permutations of n elements. The labelled constructions introduced in Definition 1.3.16 will allow us to establish a connection between this fact and an interpretation of permutations as sets of cycles.

DEFINITION 1.3.16. Let $\mathcal{A}$ be a labelled class. Then the sequence construction applied to $\mathcal{A}$ is given by

$$\mathrm{SEQ}(\mathcal{A}) = \mathcal{E} \cup \mathcal{A} \cup (\mathcal{A} \star \mathcal{A}) \cup \ldots,$$

the n-sequences are defined by

$$\mathrm{SEQ}_n(\mathcal{A}) = \mathcal{A}^{\star n},$$

the n-fold labelled product of $\mathcal{A}$ with itself.
The n-set construction applied to $\mathcal{A}$ is formally given by the quotient $\mathrm{SET}_n(\mathcal{A}) = \mathrm{SEQ}_n(\mathcal{A}) / \boldsymbol{R}$ where the equivalence relation $\boldsymbol{R}$ identifies two sequences when the components of one are a permutation of the components of the other (i.e., we neglect the ordering). The set construction can then be defined by

$$\mathrm{SET}(\mathcal{A}) = \mathcal{E} \cup \mathcal{A} \cup \mathrm{SET}_2(\mathcal{A}) \cup \ldots$$

The class of n-cycles of $\mathcal{A}$ is given by the quotient $\mathrm{CYC}_n(\mathcal{A}) = \mathrm{SEQ}_n(\mathcal{A}) / \boldsymbol{S}$ where the equivalence relation $\boldsymbol{S}$ identifies two sequences when the components of one sequence are a cyclic permutation of the components of the other. The cycle construction is given by

$$\mathrm{CYC}(\mathcal{A}) = \mathcal{E} \cup \mathcal{A} \cup \mathrm{CYC}_2(\mathcal{A}) \cup \ldots$$

THEOREM 1.3.17 ([**30**, Theorem II.1]). *Let $\mathcal{B}$ be a labelled class with EGF $B(z)$. Then the following statements hold:*

(1) If $\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$, then its EGF is given by $A(z) = \frac{1}{1-B(z)}$.
(2) If $\mathcal{A} = \mathrm{SEQ}_n(\mathcal{B})$, then $A(z) = B(z)^n$.
(3) If $\mathcal{A} = \mathrm{SET}(\mathcal{B})$, then $A(z) = \exp(B(z))$.
(4) If $\mathcal{A} = \mathrm{SET}_n(\mathcal{B})$, then $A(z) = \frac{B(z)^n}{n!}$.
(5) If $\mathcal{A} = \mathrm{CYC}(\mathcal{B})$, then $A(z) = \log\left(\frac{1}{1-B(z)}\right)$.
(6) If $\mathcal{A} = \mathrm{CYC}_n(\mathcal{B})$, then $A(z) = \frac{B(z)^n}{n}$.

PROOF. The second statement follows immediately from the respective properties of the labelled product. Recall that the labelled product always involves a relabelling. Statements (4) and (6) then additionally respect the actions of the equivalence relations $\boldsymbol{R}$ (we forget any ordering, factor of $1/n!$) and $\boldsymbol{S}$ (a sequence of length n allows n cyclic shifts, factor of $1/n$), respectively. The remaining statements follow from summing up the EGFs of the n-component constructions and applying well-known summation formulas for geometric and Taylor series. □

Example 1.3.18. Let $\mathcal{Z}$ be the labelled class containing exactly one element of size 1. Then $\mathrm{CYC}(\mathcal{Z})$ can be interpreted as the labelled class of cyclic permutations or cycles. Since any permutation can be written as a product of pairwise disjoint cycles (they therefore commute), we can identify permutations as labelled objects with $\mathrm{SET}(\mathrm{CYC}(\mathcal{Z}))$. This is consistent with

$$P(z) = \frac{1}{1-z} = \exp\left(\log\left(\frac{1}{1-z}\right)\right)$$

by Theorem 1.3.17.

Intuitively speaking, the exponential function combines cycles of different lengths to form a permutation. If we include a factor of s for any cycle appearing in any permutation, we thus arrive at the bivariate generating function

$$P_C(z,s) = \sum_{n=0}^{\infty} \sum_{\sigma \in S_n} s^{C(\sigma)} \frac{z^n}{n!} = \exp\left(s \log\left(\frac{1}{1-z}\right)\right).$$

The probability-generating function of the total number of cycles C under the uniform measure on the symmetric group S_n is then given by

$$\frac{[z^n]\, P_C(z,s)}{[z^n]\, P_C(z,1)} = \frac{1}{n!} \sum_{\sigma \in S_n} s^{C(\sigma)}.$$

Many more multivariate generating functions may be produced in this way. In particular, one might expand the logarithm and assign a specific variable s_j for each possible cycle length j or drop the summand of $\frac{z^j}{j}$ for a number of lengths j altogether. Proceeding in such a way allows us to encode the information about the cycle structure of any probabilistic model of permutations defined by cycle weights as a multivariate generating function.
The theory of MGFs and how additional parameters are inherited when applying constructions has been developped in full generality in [**30**, III.4.]. Since the results conform to natural expectations, we turn directly to the special case of permutations in Section 1.3.3, where we will also provide a rigorous proof.

1.3.3. A Shortcut for Permutations via Pólya's Enumeration Theorem. This section presents material from [**10**, Section 3] and provides the general framework for obtaining suitable (multivariate) generating functions corresponding to permutations. We also draw certain conclusions concerning random permutations without macroscopic cycles. In particular, Equation (1.3.6) will serve as an important reference point throughout this thesis.
Proposition 1.3.19 is a special case of Pólya's Enumeration Theorem (see [**53**, §16, p. 17]).

Proposition 1.3.19. *Let $(q_j)_{j \in \mathbb{N}}$ be a complex-valued sequence. Then*

$$\exp\left(\sum_{j=1}^{\infty} \frac{q_j z^j}{j}\right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\sigma \in S_n} \prod_{j=1}^{n} q_j^{C_j} \tag{1.3.2}$$

holds as an identity between formal power series in z. Here, $C_j = C_j(\sigma)$ is the number of cycles of length j in σ. If either of the series in Equation (1.3.2) is absolutely convergent, so is the other one.

Proof ([**15**]). We can regroup the permutations $\sigma \in S_n$ into subclasses according to their cycle structure $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)$ so that $C_j(\sigma) = c_j$ for all $1 \le j \le n$. If $N_{\boldsymbol{c}}$ denotes the number of permutations in S_n having the cycle structure $\boldsymbol{c}$, summation over all possible cycle structures $\boldsymbol{c}$ (i.e., $c_j \in \mathbb{N}_0$ for all j and $\sum_{j=1}^{n} j c_j = n$) yields

$$\sum_{\sigma \in S_n} \prod_{j=1}^{n} q_j^{C_j(\sigma)} = \sum_{\boldsymbol{c}} N_{\boldsymbol{c}} \prod_{j=1}^{n} q_j^{c_j}. \tag{1.3.3}$$

By Cauchy's formula (see Equation (1.1.3)), we obtain that

$$N_{\boldsymbol{c}} = \frac{n!}{\prod_{j=1}^{n} j^{c_j} c_j!}. \tag{1.3.4}$$

By inserting Equation (1.3.4) into Equation (1.3.3),

$$\sum_{\sigma\in S_n}\prod_{j=1}^{n} q_j^{C_j(\sigma)} = \sum_{\boldsymbol{c}} \frac{n!}{\prod_{j=1}^{n} j^{c_j} c_j!}\prod_{j=1}^{n} q_j^{c_j}$$

follows. Note further that $z^n = \prod_j z^{jc_j}$ for each cycle structure $\boldsymbol{c}$ occurring in S_n. Hence,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{\sigma\in S_n}\prod_{j=1}^{n} q_j^{C_j} =& 1+\sum_{n=1}^{\infty}\sum_{\boldsymbol{c}}\prod_{j=1}^{n}\frac{1}{c_j!}\left(\frac{q_j z^j}{j}\right)^{c_j}\\ =& \prod_{j=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{q_j z^j}{j}\right)^{k}\\ =& \prod_{j=1}^{\infty}\exp\left(\frac{q_j z^j}{j}\right)\\ =& \exp\left(\sum_{j=1}^{\infty}\frac{q_j z^j}{j}\right).\end{aligned}$$

The remaining claim about absolute convergence is a consequence of Lebesgue's dominated convergence theorem. □

If we extract the nth coefficient, we obtain

$$[z^n]\exp\left(\sum_{j=1}^{\infty}\frac{q_j z^j}{j}\right) = \frac{1}{n!}\sum_{\sigma\in S_n}\prod_{j=1}^{n} q_j^{C_j}. \tag{1.3.5}$$

Suitable choices of $\boldsymbol{q} = (q_j)_j$ such as $q_j = \vartheta \mathrm{e}^{s_j}$ for $\vartheta > 0$ allow us, e.g., to find a tractable expression for the moment-generating function of the joint distribution of the cycle counts under the Ewens measure with parameter ϑ on S_n.
Let us develop the general formula as it relates to random permutations without macroscopic cycles and point out some of its distinctive features. If we set $q_j = 0$ for $j > \alpha(n)$ in Equation (1.3.5), we have

$$[z^n]\exp\left(\sum_{j=1}^{\alpha(n)}\frac{q_j z^j}{j}\right) = \frac{1}{n!}\sum_{\sigma\in S_{n,\alpha}}\prod_{j=1}^{\alpha(n)} q_j^{C_j}. \tag{1.3.6}$$

Note that there is now a dependence of the function $\exp\left(\sum_{j=1}^{\alpha(n)}\frac{q_j z^j}{j}\right)$ on n, which complicates matters considerably since many classical techniques which investigate the asymptotics of the coefficients of a fixed function (cf. Section 1.3.4) no longer work. In this thesis we will therefore draw upon the saddle-point method (see Section 1.3.5), which can be applied to the case of random permutations without macroscopic cycles. We build on **[45]** where the asymptotics of the local probabilities

$$\mathbb{P}_n^{(1)}[S_{n,\alpha}] = [z^n]\exp\left(\sum_{j=1}^{\alpha(n)}\frac{z^j}{j}\right)$$

are calculated and extend the results in Section 2.1.

1.3.4. Singularity Analysis, Tauberian Theory, and Darboux's Method. Having presented methods to obtain generating functions related to specific problems, the next step is extracting the relevant coefficients. One important such technique is singularity analysis. We assume that $A(z) = \sum_{n=0}^{\infty} A_n z^n$ is a generating function which is analytic on a suitable domain containing 0 and has a, for the sake of simplicity, single singularity at $z_0 \neq 0$. Our goal is to understand the asymptotics of A_n as $n\to\infty$. The key insight can be summarized as follows: There is a correspondence between the asymptotic behaviour of an analytic function $A(z)$ near its (dominant) singularity and the asymptotics of the coefficients A_n in its expansion.

The method of singularity analysis applies to functions from the so-called standard function scale whose singularities are asymptotically of the type of fractional powers and logarithms, a typical function of this class satisfying

$$A(z) \sim \left(1 - \frac{z}{z_0}\right)^{-a} \left(\log\left(\frac{1}{1-\frac{z}{z_0}}\right)\right)^b \tag{1.3.7}$$

as z tends to the singularity z_0 for $a, b \in \mathbb{C}$. Its coefficients are then asymptotically of the form

$$A_n = [z^n] A(z) \sim z_0^{-n} n^{a-1} (\log n)^b .$$

A singular expansion at z_0 is given by

$$A(z) = \sigma(z) + \mathcal{O}(\tau(z)), \tag{1.3.8}$$

where $\tau(z) = o(\sigma(z))$ as $z \to z_0$ and both σ and τ belong to the standard function scale. Singularity analysis of the function $A(z)$ then involves two ingredients: Firstly, there is a list of asymptotic expansions of a set of standard functions (containing σ and τ). Secondly, we need a transfer theorem which allows us to conclude

$$A_n = \sigma_n + \mathcal{O}(\tau_n)$$

from Equation (1.3.8). Here, $\sigma_n = [z^n]\sigma(z)$ and $\tau_n = [z^n]\tau(z)$. The proofs of these methods are based on Cauchy's integral formula and integration along special contours (cf. [**30**, Figure VI.2]), among them the Hankel contour known from Hankel's representation of the gamma function (see [**30**, Section B.3]). They additionally require analytic continuation so that the contour integrals are well-defined. See [**30**, Theorem VI.4] for a precise statement. Note that the range of behaviour at the singularity to which singularity analysis applies is much richer than only the one given by Equation (1.3.7). Moreover, the class of functions from the standard function scale is closed under operations such as sum and product which are key tools for the symbolic method (cf. Section 1.3.2).

Classical alternatives to singularity analysis include Tauberian theory and Darboux's method. Whereas singularity analysis has to presuppose analytic continuation, Tauberian theory requires positivity or monotonicity of the coefficients and Darboux's method assumes certain differentiability conditions of the function on its circle of convergence. A famous Tauberian theorem due to Hardy, Littlewood, and Karamata is

THEOREM 1.3.20 (The HLK Tauberian Theorem, [**30**, Theorem VI.13]). *Let $A(z)$ be a power series with radius of convergence equal to* 1*, satisfying*

$$A(z) \sim \frac{1}{(1-z)^a} \Lambda\left(\frac{1}{1-z}\right)$$

for some $a > 0$, where Λ is a slowly varying function. Assume further that the coefficients A_n are non-negative. Then,

$$\sum_{n=0}^{N} A_n \sim \frac{N^a}{\Gamma(a+1)} \Lambda(N)$$

as $N \to \infty$.

Note that the assumptions of Theorem 1.3.20 are mild, but it does not provide error estimates. Darboux's method establishes a correspondence between the smoothness of a function and the decay of its coefficients.

THEOREM 1.3.21 (Darboux's method, [**30**, Theorem VI.14]). *Assume that $A(z)$ is continuous in the closed unit disc in $\mathbb{C}$ and is, in addition, $K \geq 0$ times continously differentiable on $\partial B_1(0)$. Then,*

$$A_n = o\left(n^{-K}\right)$$

as $n \to \infty$.

It is a built-in feature of Darboux's method that it cannot be applied to singular expansions that only involve diverging terms. But there are also examples to which Darboux's method applies and singularity analysis does not (see [**30**, VI.32]).

All methods presented in this section deal with the asymptotics of $[z^n] A(z)$ for a fixed function $A(z)$. As Equation (1.3.6) shows, relevant problems concerning random permutations without macroscopic cycles do not fall into this category since their study involves the asymptotics of expressions such as $[z^n] A_n(z)$, which feature a different analytic function $A_n(z)$ for each $n \in \mathbb{N}$. Certain proof techniques employed in singularity analysis may, however, still apply if the dependence of $A_n(z)$ on n is sufficiently explicit (see Sections 3.1.2 and 3.1.3). This thesis relies on a different method which is up to the task of treating random permutations without macroscopic cycles. It will be presented in the next section.

1.3.5. Saddle-Point Method. The saddle-point method is a complex analogue of Laplace's method and a powerful tool for approximating contour integrals which arise in analytic combinatorics. Let $h(z)$ be an analytic function and consider $H_n(z) = \exp(nh(z))$. A saddle point z_0 of $H_n(z)$ is then defined by

$$\frac{\mathrm{d}}{\mathrm{d}z} H_n(z_0) = 0$$

or, equivalently,

$$nh'(z_0) = 0. \tag{1.3.9}$$

Note that Equation (1.3.9) allows us to approximate $h(z)$ by a quadratic function in a neighbourhood of z_0. The central idea of the saddle-point method is the following: Given a contour γ containing z_0 such that $\Re h(z)$ adopts a local maximum in z_0, the integral

$$I_n := \int_\gamma \mathrm{e}^{nh(z)} \mathrm{d}z$$

will, in the limit of large n, be dominated by the contribution from a small part γ_1 of the contour γ which is located in a neighbourhood of z_0:

$$I_n = \int_{\gamma_1} \mathrm{e}^{nh(z)} \mathrm{d}z + \int_{\gamma_2} \mathrm{e}^{nh(z)} \mathrm{d}z \sim \int_{\gamma_1} \mathrm{e}^{nh(z)} \mathrm{d}z.$$

Here, γ_2 denotes the rest of the contour γ. If, in addition, the quadratic approximation of $h(z)$ is valid along the contour γ_1, the integral $\int_{\gamma_1} \mathrm{e}^{nh(z)} \mathrm{d}z$ can be approximated by an incomplete Gaussian integral. In cases where the Gaussian tails are of lower order, we obtain that I_n is asymptotically equivalent to a complete Gaussian integral for which there are closed forms.
Applying the saddle-point method thus requires a (single) saddle point and a suitable contour. The concrete proof then consists of three steps (see [**30**, Figure VIII.4]):

(1) Tails pruning: The integral $\int_{\gamma_2} \mathrm{e}^{nh(z)} \mathrm{d}z$ is of lower order than I_n.
(2) Central approximation: Along γ_1, a quadratic expansion of $h(z)$ holds.
(3) Tails completion: $\int_{\gamma_1} \mathrm{e}^{nh(z)} \mathrm{d}z$ is asymptotically equivalent to a complete Gaussian integral.

Note that the saddle-point method may also be applied to more general functions $h_n(z)$ instead of $nh(z)$, where the dependence on the parameter n is less explicit. If we consider the special case of asymptotics of coefficients of generating functions $A_n(z)$, more can be said. Assume for each $n \in \mathbb{N}_0$ that $A_n(z)$ is analytic at the origin and that its coefficients are non-negative. We want to compute

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma A_n(z) \frac{\mathrm{d}z}{z^{n+1}}.$$

A saddle point has to satisfy

$$0 = \left(\frac{A_n(z)}{z^{n+1}} \right)',$$

which is equivalent to

$$n + 1 = z \frac{A_n'(z)}{A_n(z)}. \tag{1.3.10}$$

Equation (1.3.10) has a unique (cf. [**30**, VIII.4]) positive solution. We thus choose the contour $\gamma(\varphi) = x\mathrm{e}^{\mathrm{i}\varphi}$ for $-\pi < \varphi < \pi$ and $x > 0$. It follows that

$$\frac{1}{2\pi\mathrm{i}} \int_{\gamma} A_n(z) \frac{\mathrm{d}z}{z^{n+1}} = \frac{x^{-n}}{2\pi} \int_{-\pi}^{\pi} A_n\left(x\mathrm{e}^{\mathrm{i}\varphi}\right) \mathrm{e}^{-\mathrm{i}n\varphi} \mathrm{d}\varphi. \tag{1.3.11}$$

By differentiating $\log(A_n(z)) - \mathrm{i}n\log(z)$ and setting its value to zero, we obtain an alternative saddle-point equation related to polar coordinates,

$$n = x\frac{A_n'(x)}{A_n(x)}, \tag{1.3.12}$$

which is the version we will use in Section 2.1 since there is no linear term in the logarithm of the integrand in Equation (1.3.11) with this choice. Note that Equations (1.3.10) and (1.3.12) are very similar for large n.

Applying the saddle-point method now involves identifying a cut-off angle φ_n such that tails pruning, central approximation, and tails completion may be verified with

$$\frac{1}{2\pi\mathrm{i}} \int_{\gamma_1} A_n(z) \frac{\mathrm{d}z}{z^{n+1}} = \frac{x^{-n}}{2\pi} \int_{-\varphi_n}^{\varphi_n} A_n\left(x\mathrm{e}^{\mathrm{i}\varphi}\right) \mathrm{e}^{-\mathrm{i}n\varphi} \mathrm{d}\varphi.$$

An example of this case, which is of vital importance for this thesis and additionally introduces certain perturbations, is treated in detail in Proposition 2.1.4.

CHAPTER 2

Random Permutations without Macroscopic Cycles

In this chapter we give the results concerning constrained random permutations. Recall that, for $\alpha : \mathbb{N} \to \mathbb{N}$ such that $1 \leq \alpha(n) \leq n$ for all n, we define the set

$$S_{n,\alpha} := \{\sigma \in S_n | \text{no cycle in } \sigma \text{ is longer than } \alpha(n)\}$$

of permutations without long cycles. Then, for $\vartheta > 0$, $\mathbb{P}^{(\vartheta)}_{n,\alpha}[\cdot] := \mathbb{P}^{(\vartheta)}_n[\cdot | S_{n,\alpha}]$ is a probability measure concentrated on $S_{n,\alpha}$ which we call random permutations without macroscopic cycles if $\alpha(n) = o(n)$ holds, i.e. $\lim_{n\to\infty} \alpha(n)/n = 0$. Depending on the value of ϑ, $\mathbb{P}^{(\vartheta)}_{n,\alpha}$ is also called the constrained Ewens or uniform measure. We have

$$\mathbb{P}^{(\vartheta)}_{n,\alpha}[\{\sigma\}] = \frac{1}{Z_{n,\alpha,\vartheta}} \frac{1}{n!} \vartheta^{C(\sigma)}$$

for $\sigma \in S_{n,\alpha}$, where the normalizing constant is given by

$$Z_{n,\alpha,\vartheta} = \frac{1}{n!} \sum_{\sigma \in S_{n,\alpha}} \vartheta^{C(\sigma)}. \tag{2.0.1}$$

In most of the following sections, we will assume that the sequence α satisfies

$$n^{a_1} \leq \alpha(n) \leq n^{a_2} \tag{2.0.2}$$

for fixed numbers $a_1, a_2 \in (0,1)$ and for all $n \in \mathbb{N}$.

2.1. The Saddle-Point Method

The following Proposition 2.1.4 presents a version of the saddle-point method (see Section 1.3.5 for an introduction) which will be sufficient for most applications in this thesis. Let

$$\boldsymbol{q} = (q_{j,n})_{1 \leq j \leq \alpha(n), n \in \mathbb{N}}$$

be a triangular array. Due to the problems under consideration, we may assume that all $q_{j,n}$ are non-negative. Let then the saddle point $x_{n,\boldsymbol{q}}$ be given as the unique positive solution of

$$n = \sum_{j=1}^{\alpha(n)} q_{j,n} x^j_{n,\boldsymbol{q}}. \tag{2.1.1}$$

Equation (2.1.1) is to be interpreted in light of Equation (1.3.12) with respect to Equation (2.1.6) below. We additionally define

$$\lambda_{p,n} := \lambda_{p,n,\alpha,\boldsymbol{q}} := \sum_{j=1}^{\alpha(n)} q_{j,n} j^{p-1} x^j_{n,\boldsymbol{q}} \tag{2.1.2}$$

for natural numbers p. Note that $n = \lambda_{1,n}$ is a reformulation of Equation (2.1.1), whence we conclude that

$$\lambda_{p,n} \leq n (\alpha(n))^{p-1} \tag{2.1.3}$$

for $p \geq 1$.

We collect a few concepts which will be of importance throughout the thesis. We say that

$$f_n(t) = \mathcal{O}(g_n(t))$$

pointwise in t as $n \to \infty$ if for any t there are constants C_t, N_t such that

$$|f_n(t)| \leq C_t g_n(t)$$

for all $n \geq N_t$. We have

$$f_n(t) = \mathcal{O}(g_n(t))$$

uniformly in $t \in T_n$ as $n \to \infty$ if there are constants $C, N > 0$ such that

$$\sup_{t \in T_n} \left\{ \left| \frac{f_n(t)}{g_n(t)} \right| \right\} \leq C$$

holds for all $n \geq N$. In an analogous manner, let

$$f_n(t) \sim g_n(t)$$

pointwise in t as $n \to \infty$ if for any t we have

$$\lim_{n \to \infty} \frac{f_n(t)}{g_n(t)} = 1$$

and

$$f_n(t) \sim g_n(t)$$

uniformly in $t \in T_n$ as $n \to \infty$ if

$$\sup_{t \in T_n} \left\{ \left| \frac{f_n(t)}{g_n(t)} - 1 \right| \right\} \xrightarrow{n \to \infty} 0.$$

We also define

$$f_n(t) = o(g_n(t))$$

pointwise in t as $n \to \infty$ if for any t we have

$$\lim_{n \to \infty} \frac{f_n(t)}{g_n(t)} = 0$$

and

$$f_n(t) = o(g_n(t))$$

uniformly in $t \in T_n$ as $n \to \infty$ if

$$\lim_{n \to \infty} \sup_{t \in T_n} \left\{ \left| \frac{f_n(t)}{g_n(t)} \right| \right\} = 0.$$

We will sometimes drop certain specifications if they are clear from the context. This applies in particular to "as $n \to \infty$".
Let us further write $a_n \approx b_n$ for two real-valued sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ if there are constants $c, C > 0$ such that

$$c\, b_n \leq a_n \leq C\, b_n$$

holds for large n.
Definitions 2.1.1 and 2.1.2 assemble the concepts necessary to formulate our version of the saddle-point method in a concise way.

DEFINITION 2.1.1. A triangular array $\boldsymbol{q}$ is admissible if the following three conditions hold:

(1) It satisfies

$$\alpha(n) \log(x_{n,\boldsymbol{q}}) \approx \log\left(\frac{n}{\alpha(n)}\right).$$

In particular, $x_{n,\boldsymbol{q}} > 1$ for n large enough.

(2) We have

$$\lambda_{2,n} \approx n\alpha(n).$$

(3) There exist a non-negative sequence $(b_n)_{n \in \mathbb{N}}$ and constants $\delta, c > 0$ such that $b(n)/\alpha(n) < 1 - \delta$ and $q_{j,n} \geq c > 0$ for all $j \geq b(n)$ hold for n large enough.

DEFINITION 2.1.2. Let $\boldsymbol{q}$ be a triangular array. Then a sequence $(f_n)_{n \in \mathbb{N}}$ of functions is called admissible (w.r.t. $\boldsymbol{q}$) if it satisfies the following three conditions:

(1) There is $\delta > 0$ such that f_n is holomorphic on the disc $B_{x_{n,\boldsymbol{q}}+\delta}(0)$ if $n \in \mathbb{N}$ is large enough.

(2) There exist constants $K, N > 0$ such that

$$\sup_{z\in\partial B_{x_{n,\boldsymbol{q}}}(0)} |f_n(z)| \leq n^K |f_n(x_{n,\boldsymbol{q}})| \tag{2.1.4}$$

for all $n \geq N$.

(3) With the definition

$$\|\!|f_n\|\!|_n := n^{-\frac{5}{12}} (\alpha(n))^{-\frac{7}{12}} \sup_{|\varphi|\leq n^{-\frac{5}{12}}(\alpha(n))^{-\frac{7}{12}}} \frac{|f_n'(x_{n,\boldsymbol{q}}e^{i\varphi})|}{|f_n(x_{n,\boldsymbol{q}})|}, \tag{2.1.5}$$

we have $\|\!|f_n\|\!|_n \to 0$ as $n \to \infty$.

REMARK 2.1.3. The conditions in Definitions 2.1.1 and 2.1.2 are formulated in light of the proof of Proposition 2.1.4 below and the applications in later sections. Conditions (1) and (2) in Definition 2.1.1 will follow naturally for the model of random permutations without macroscopic cycles (see Sections 2.2 and 2.3), whereas condition (3) is tailor-made to suit the needs of Section 2.5 (see, specifically, the proof of Lemma 2.5.6) without unduly interfering with other cases. Certain alternatives will be discussed in Section 3.2. Definition 2.1.2 gives sufficient conditions for perturbations f_n to be small enough such that the underlying saddle-point method still works. Condition (1) is very natural and will allow us to use Cauchy's integral formula with contours suited for the saddle-point method. Intuitively, condition (2) ensures that a small neighbourhood of the saddle point still provides the dominant contribution (so tails pruning is not affected). Condition (3) is the most restrictive for applications since there is not much leeway to suppress (relatively) large derivatives, which occur naturally in many instances. It is intended to keep the perturbations compatible with the step of central approximation.

In [**45**], the saddle-point method is applied in order to count the total number of permutations without long cycles for a large range of sequences α, namely for α such that $4 \leq \alpha(n) \leq (12\pi^2 e)^{-1} n (\log(n) \log(\log(n)))^{-1}$. On the one hand, the following Proposition 2.1.4 treats only sequences α which grow algebraically in n (cf. Equation (2.0.2)), but, on the other hand, it allows us to introduce cycle weights and perturb the original saddle-point method. We are thus enabled to obtain the asymptotics for a variety of moment-generating functions of interest. The restriction to consider only algebraically growing sequences α facilitates the treatment of perturbations in this context, but improvements are possible (see Section 3.2.1). In this sense, Proposition 2.1.4 extends the results and methods developped in [**45**], on which its proof rests.

PROPOSITION 2.1.4 ([**10**, Proposition 3.2]). *Assume that $\alpha(n)$ satisfies Equation (2.0.2). Let $\boldsymbol{q}$ be an admissible triangular array and $(f_n)_{n\in\mathbb{N}}$ an admissible sequence of functions. Then we have*

$$[z^n] f_n(z) \exp\left(\sum_{j=1}^{\alpha(n)} \frac{q_{j,n}}{j} z^j\right) = f_n(x_{n,\boldsymbol{q}}) \frac{\exp(\lambda_{0,n})}{x_{n,\boldsymbol{q}}^n \sqrt{2\pi\lambda_{2,n}}} \left(1 + \mathcal{O}\left(\frac{\alpha(n)}{n} + \|\!|f_n\|\!|_n\right)\right)$$

for large n.

REMARK 2.1.5. As the proof will show, if an array $\boldsymbol{q}$ has been fixed, the error term $\mathcal{O}\left(\frac{\alpha(n)}{n} + \|\!|f_n\|\!|_n\right)$ in Proposition 2.1.4 is uniform in sequences of functions $(f_n)_n$ which satisfy condition (2) in Definition 2.1.2 for the same constants K and N.

PROOF OF PROPOSITION 2.1.4. Cauchy's integral formula yields

$$M_n := [z^n] f_n(z) \exp\left(\sum_{j=1}^{\alpha(n)} \frac{q_{j,n}}{j} z^j\right) = \frac{1}{2\pi i} \int_{\partial B_r(0)} f_n(z) \exp\left(\sum_{j=1}^{\alpha(n)} \frac{q_{j,n}}{j} z^j\right) \frac{dz}{z^{n+1}}$$

for any $r > 0$ such that f_n is holomorphic on an open disc $B_R(0)$ with $R > r$. Since $(f_n)_{n\in\mathbb{N}}$ is admissible, condition (1) in Definition 2.1.2 allows us to choose $r = x_{n,\boldsymbol{q}}$ if n is large enough. For the sake of brevity, we will write x for $x_{n,\boldsymbol{q}}$ for the rest of the proof. We then obtain

$$M_n = \frac{1}{2\pi x^n} \int_{-\pi}^{\pi} f_n(xe^{i\varphi}) \exp\left(\sum_{j=1}^{\alpha(n)} \frac{q_{j,n}}{j} x^j e^{ij\varphi} - in\varphi\right) d\varphi. \tag{2.1.6}$$

With the definition

$$g_n(\varphi) := \sum_{j=1}^{\alpha(n)} q_{j,n} \frac{\mathrm{e}^{\mathrm{i}j\varphi} - 1}{j} x^j - \mathrm{i}n\varphi,$$

we arrive at

$$M_n = \frac{\exp\left(\sum_{j=1}^{\alpha(n)} \frac{q_{j,n}}{j} x^j\right)}{2\pi x^n} \int_{-\pi}^{\pi} f_n\left(x\mathrm{e}^{\mathrm{i}\varphi}\right) \exp\left(g_n(\varphi)\right) \mathrm{d}\varphi.$$

Note the immediate relations $g_n(0) = g_n'(0) = 0$ (x is the saddle point and therefore satisfies Equation (2.1.1)), $g_n^{(p)}(0) = \mathrm{i}^p \lambda_{p,n}$, and $\left|g_n^{(p)}(\varphi)\right| \leq \lambda_{p,n}$ for all φ and $p > 1$ (both due to Equation (2.1.2)).

We follow the playbook described in Section 1.3.5 and start with

Central Approximation:

The first step in the proof is thus to consider a small part of the integral about $\varphi = 0$: Let $\varphi_n := n^{-\frac{5}{12}} (\alpha(n))^{-\frac{7}{12}}$. Then $\lambda_{p,n} |\varphi|^p \leq n (\alpha(n))^{p-1} n^{-\frac{5p}{12}} (\alpha(n))^{-\frac{7p}{12}} \leq \left(\frac{n}{\alpha(n)}\right)^{1-\frac{5p}{12}}$ for all $|\varphi| \leq \varphi_n$ due to Equation (2.1.3). Expanding g_n about 0 thus yields

$$g_n(\varphi) = -\frac{\lambda_{2,n}}{2}\varphi^2 - \mathrm{i}\frac{\lambda_{3,n}}{6}\varphi^3 + \mathcal{O}\left(\lambda_{4,n}\varphi^4\right)$$

uniformly in $|\varphi| \leq \varphi_n$. Since $\lambda_{4,n}\varphi^4 \leq \left(\frac{n}{\alpha(n)}\right)^{-\frac{2}{3}} = o(1)$ for $|\varphi| \leq \varphi_n$, expanding the exponential function leads to

$$\exp\left(g_n(\varphi)\right) = \exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2 - \mathrm{i}\frac{\lambda_{3,n}}{6}\varphi^3\right)\left(1 + \mathcal{O}\left(\lambda_{4,n}\varphi^4\right)\right)$$

uniformly in $|\varphi| \leq \varphi_n$. Similarly, we obtain

$$\exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2 - \mathrm{i}\frac{\lambda_{3,n}}{6}\varphi^3\right) = \exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right)\left(1 - \mathrm{i}\frac{\lambda_{3,n}}{6}\varphi^3 + \mathcal{O}\left(\lambda_{3,n}^2\varphi^6\right)\right)$$

uniformly in $|\varphi| \leq \varphi_n$ and therefore conclude that

$$\exp\left(g_n(\varphi)\right) = \exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right)\left(1 - \mathrm{i}\frac{\lambda_{3,n}}{6}\varphi^3 + \mathcal{O}\left(\lambda_{3,n}^2\varphi^6\right)\right)\left(1 + \mathcal{O}\left(\lambda_{4,n}\varphi^4\right)\right) \tag{2.1.7}$$

uniformly in $|\varphi| \leq \varphi_n$. By admissibility, since f_n is holomorphic on $B_{x+\delta}$ for large n (see condition (1) in Definition 2.1.2), we have

$$f_n\left(x\mathrm{e}^{\mathrm{i}\varphi}\right) = f_n(x) + \int_{\gamma_\varphi} f_n'(z)\,\mathrm{d}z = f_n(x) + \mathrm{i}\int_0^{\varphi} f_n'\left(x\mathrm{e}^{\mathrm{i}\phi}\right) x\mathrm{e}^{\mathrm{i}\phi}\mathrm{d}\phi.$$

Here, γ_φ is a contour which connects the two points x and $x\mathrm{e}^{\mathrm{i}\varphi}$ along the circle $\partial B_x(0)$. By condition (3) in Definition 2.1.2, we have

$$\left|\int_0^{\varphi} f_n'\left(x\mathrm{e}^{\mathrm{i}\phi}\right) x\mathrm{e}^{\mathrm{i}\phi}\mathrm{d}\phi\right| \leq \int_0^{\varphi} \left|f_n'\left(x\mathrm{e}^{\mathrm{i}\phi}\right) x\mathrm{e}^{\mathrm{i}\phi}\right| \mathrm{d}\phi \leq x\,|f_n(x)|\,\|\!|f_n|\!\|_n$$

for all $|\varphi| \leq \varphi_n$. Since condition (1) in Definition 2.1.1 entails in particular that $\lim_{n\to\infty} x = 1$,

$$f_n\left(x\mathrm{e}^{\mathrm{i}\varphi}\right) = f_n(x)\left(1 + \mathcal{O}\left(\|\!|f_n|\!\|_n\right)\right) \tag{2.1.8}$$

holds uniformly in $|\varphi| \leq \varphi_n$. Note $\mathrm{i}\frac{\lambda_{3,n}}{6}\varphi^3 = o(1)$ uniformly in $|\varphi| \leq \varphi_n$ and that it is also an odd function. So

$$\int_{-\varphi_n}^{\varphi_n} \exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right)\mathrm{i}\frac{\lambda_{3,n}}{6}\varphi^3\mathrm{d}\varphi = 0$$

for all n, and Equations (2.1.7) and (2.1.8) then lead to

$$\begin{aligned}\int_{-\varphi_n}^{\varphi_n} f_n\left(x\mathrm{e}^{\mathrm{i}\varphi}\right)\exp\left(g_n(\varphi)\right)\mathrm{d}\varphi =& f_n(x)\int_{-\varphi_n}^{\varphi_n} \mathrm{e}^{-\frac{\lambda_{2,n}}{2}\varphi^2}\left(1 + \mathcal{O}\left(\lambda_{3,n}^2\varphi^6 + \lambda_{4,n}\varphi^4\right)\right)\mathrm{d}\varphi \\ &+ f_n(x)\int_{-\varphi_n}^{\varphi_n} \mathrm{e}^{-\frac{\lambda_{2,n}}{2}\varphi^2}\mathcal{O}\left(\|\!|f_n|\!\|_n\right)\mathrm{d}\varphi\end{aligned}$$

uniformly in $|\varphi| \leq \varphi_n$. We can now turn to the question of

Tails Completion:
Condition (2) in Definition 2.1.1 implies that $\sqrt{\lambda_{2,n}}\varphi_n \approx n^{\frac{1}{12}} (\alpha(n))^{-\frac{1}{12}} \to \infty$ as $n \to \infty$. We can therefore asymptotically bound the Gaussian tails by

$$\begin{aligned}\int_{|\varphi|>\varphi_n} \exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right) \mathrm{d}\varphi =& \frac{1}{\sqrt{\lambda_{2,n}}} \int_{|\theta|>\sqrt{\lambda_{2,n}}\varphi_n} \exp\left(-\frac{\theta^2}{2}\right) \mathrm{d}\theta \\ =& \mathcal{O}\left(\frac{\exp\left(-c_0\left(\frac{n}{\alpha(n)}\right)^{1/6}\right)}{n^{7/12}(\alpha(n))^{5/12}}\right)\end{aligned} \tag{2.1.9}$$

for some $c_0 > 0$ due to [**21**, Eq. 7.8.2]. If we apply this fact to

$$\int_{-\varphi_n}^{\varphi_n} \exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right) \mathrm{d}\varphi = \int_{-\infty}^{\infty} \exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right) \mathrm{d}\varphi - \int_{|\varphi|>\varphi_n} \exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right) \mathrm{d}\varphi,$$

we obtain in particular that

$$\int_{-\varphi_n}^{\varphi_n} \exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right) \mathrm{d}\varphi = \sqrt{\frac{2\pi}{\lambda_{2,n}}}\left(1 + \mathcal{O}\left(\frac{\alpha(n)}{n}\right)\right).$$

The same substitution $\theta = \sqrt{\lambda_{2,n}}\varphi$ together with Equation (2.1.3), condition (2) in Definition 2.1.1, and the moments of the standard normal distribution also entails

$$\int_{-\varphi_n}^{\varphi_n} \mathrm{e}^{-\frac{\lambda_{2,n}}{2}\varphi^2} \lambda_{3,n}^2 |\varphi|^6 \mathrm{d}\varphi \le 15\sqrt{\frac{2\pi}{\lambda_{2,n}}} \frac{\lambda_{3,n}^2}{\lambda_{2,n}^3} = \sqrt{\frac{2\pi}{\lambda_{2,n}}}\mathcal{O}\left(\frac{\alpha(n)}{n}\right)$$

and

$$\int_{-\varphi_n}^{\varphi_n} \mathrm{e}^{-\frac{\lambda_{2,n}}{2}\varphi^2} \lambda_{4,n} |\varphi|^4 \mathrm{d}\varphi \le 3\sqrt{\frac{2\pi}{\lambda_{2,n}}} \frac{\lambda_{4,n}}{\lambda_{2,n}^2} = \sqrt{\frac{2\pi}{\lambda_{2,n}}}\mathcal{O}\left(\frac{\alpha(n)}{n}\right).$$

The results combined then lead to

$$\int_{-\varphi_n}^{\varphi_n} f_n\left(x\mathrm{e}^{\mathrm{i}\varphi}\right) \exp\left(g_n(\varphi)\right) \mathrm{d}\varphi = f_n(x)\sqrt{\frac{2\pi}{\lambda_{2,n}}}\left(1 + \mathcal{O}\left(\frac{\alpha(n)}{n} + |||f_n|||_n\right)\right).$$

The last step is to implement

Tails Pruning:
We therefore have to show that

$$\int_{\varphi_n<|\varphi|\le\pi} f_n\left(x\mathrm{e}^{\mathrm{i}\varphi}\right) \exp\left(g_n(\varphi)\right) \mathrm{d}\varphi = \mathcal{O}\left(\frac{f_n(x)}{\sqrt{\lambda_{2,n}}}\frac{\alpha(n)}{n}\right),$$

whence the claim follows. By definition,

$$-\Re g_n(\varphi) = \sum_{j=1}^{\alpha(n)} \frac{q_{j,n}}{j}\left(1 - \cos(j\varphi)\right) x^j.$$

Let first $\varphi_n < \varphi < \frac{\pi}{\alpha(n)}$. Because of $-\partial_\varphi \Re g_n(\varphi) = \sum_{j=1}^{\alpha(n)} q_{j,n} \sin(j\varphi) x^j > 0$ and condition (2) in Definition 2.1.1,

$$-\Re g_n(\varphi) \ge -\Re g_n(\varphi_n) \approx \lambda_{2,n}\varphi_n^2 \approx \left(\frac{n}{\alpha(n)}\right)^{\frac{1}{6}} \tag{2.1.10}$$

holds. Let now $\frac{\pi}{\alpha(n)} \le \varphi \le \pi$. We first consider the case that $q_{j,n} \ge c > 0$ for all n and j. This assumption corresponds to $b(n) = 1$ in condition (3) in Definition 2.1.1. The case of general $b(n)$ will be treated afterwards. By assumption,

$$-\Re g_n(\varphi) = \sum_{j=1}^{\alpha(n)} \frac{q_{j,n}}{j}\left(1 - \cos(j\varphi)\right) x^j \ge \frac{c}{\alpha(n)} \sum_{j=1}^{\alpha(n)} \left(1 - \cos(j\varphi)\right) x^j.$$

If we define

$$r_n(\varphi) := \frac{1}{\alpha(n)} \sum_{j=1}^{\alpha(n)} (1-\cos(j\varphi))\, x^j$$

and apply the formula for the geometric series separately to $\sum_{j=1}^{\alpha(n)} x^j$ and $\sum_{j=1}^{\alpha(n)} \cos(j\varphi)\, x^j = \Re \sum_{j=1}^{\alpha(n)} \mathrm{e}^{\mathrm{i}j\varphi} x^j$, we obtain

$$r_n(\varphi) = \frac{1}{\alpha(n)} \left(x \frac{x^{\alpha(n)}-1}{x-1} - \Re x \mathrm{e}^{\mathrm{i}\varphi} \frac{x^{\alpha(n)} \mathrm{e}^{\mathrm{i}\alpha(n)\varphi}-1}{x\mathrm{e}^{\mathrm{i}\varphi}-1} \right).$$

Since the real part is bounded by the modulus,

$$\Re x \mathrm{e}^{\mathrm{i}\varphi} \frac{x^{\alpha(n)} \mathrm{e}^{\mathrm{i}\alpha(n)\varphi}-1}{x\mathrm{e}^{\mathrm{i}\varphi}-1} \leq \frac{x^{\alpha(n)+1}+x}{x-1} \frac{x-1}{|x\mathrm{e}^{\mathrm{i}\varphi}-1|}$$

follows. Consider

$$\frac{x-1}{|x\mathrm{e}^{\mathrm{i}\varphi}-1|} = \frac{x-1}{\sqrt{x^2\sin^2(\varphi) + (1-x\cos(\varphi))^2}} \leq \frac{x-1}{\sqrt{(x-1)^2 + 2(1-\cos(\varphi))}},$$

where the inequality applies $x>1$. Since $|\varphi| \leq \pi$, one easily sees that

$$\frac{2\varphi^2}{\pi^2} \leq 1 - \cos(\varphi) \leq \frac{\varphi^2}{2}.$$

We therefore have

$$\frac{x-1}{|x\mathrm{e}^{\mathrm{i}\varphi}-1|} \leq \frac{x-1}{\sqrt{(x-1)^2 + 4\varphi^2/\pi^2}}.$$

The second binomial formula entails that

$$\frac{v}{\sqrt{v^2+w^2}} \leq 1 - \frac{1}{2} \frac{w^2}{v^2+w^2}$$

for $v>0$ and $w \in \mathbb{R}$. So, with $v = x-1$ and $w = 2\varphi/\pi$, we obtain

$$\frac{x-1}{|x\mathrm{e}^{\mathrm{i}\varphi}-1|} \leq 1 - \frac{1}{2} \frac{4\varphi^2/\pi^2}{(x-1)^2 + 4\varphi^2/\pi^2}.$$

Altogether we arrive at

$$r_n(\varphi) \geq \frac{2}{\pi^2} \frac{x^{\alpha(n)+1}+x}{\alpha(n)(x-1)} \frac{\varphi^2}{(x-1)^2 + 4\varphi^2/\pi^2} - \frac{2x}{\alpha(n)(x-1)}. \tag{2.1.11}$$

By condition (1) in Definition 2.1.1 there are constants $c_1, c_2 > 0$ such that

$$c_1 \log\left(\frac{n}{\alpha(n)}\right) \leq \alpha(n) \log(x) \leq c_2 \log\left(\frac{n}{\alpha(n)}\right) \tag{2.1.12}$$

for large n. In particular, we have $x \sim 1$ and $x - 1 \sim \log(x) \geq \frac{c_1}{\alpha(n)} \log\left(\frac{n}{\alpha(n)}\right)$ as $n \to \infty$. The term

$$\frac{2x}{\alpha(n)(x-1)}$$

in Equation (2.1.11) thus converges to 0. As to the first term on the right-hand side in Equation (2.1.11), note that $\frac{\varphi^2}{(x-1)^2+4\varphi^2/\pi^2}$ is increasing in φ. So, for large n, we obtain

$$\begin{aligned}
r_n(\varphi) \geq & \frac{2}{\pi^2} \frac{x^{\alpha(n)+1}+x}{\alpha(n)\log(x)} \frac{\log(x)}{x-1} \frac{\pi^2(\alpha(n))^{-2}}{(x-1)^2 + 4(\alpha(n))^{-2}} - \frac{2x}{\alpha(n)(x-1)} \\
\geq & \frac{2}{c_2} \frac{x^{\alpha(n)+1}+x}{\log(n/\alpha(n))} \frac{\log(x)}{x-1} \frac{1}{(\alpha(n)\log(x))^2((x-1)/\log(x))^2 + 4} - \frac{2x}{\alpha(n)(x-1)} \\
\geq & \frac{2}{c_2} \frac{x^{\alpha(n)+1}+x}{\log(n/\alpha(n))} \frac{\log(x)}{x-1} \frac{1}{(c_2\log(n/\alpha(n)))^2((x-1)/\log(x))^2 + 4} - \frac{2x}{\alpha(n)(x-1)} \\
\sim & \frac{2}{c_2^3} \frac{x^{\alpha(n)+1}}{(\log(n/\alpha(n)))^3}
\end{aligned} \tag{2.1.13}$$

since $\frac{\pi}{\alpha(n)} \leq \varphi \leq \pi$. Note that $x^{\alpha(n)} \geq \left(\frac{n}{\alpha(n)}\right)^{c_1}$. By condition (2) in Definition 2.1.2 as well as Equations (2.1.10) and (2.1.13), it now follows that

$$\frac{\left|f_n\left(x\mathrm{e}^{\mathrm{i}\varphi}\right)\exp\left(g_n\left(\varphi\right)\right)\right|}{\left|f_n\left(x\right)\right|} \leq n^K\left|\exp\left(g_n\left(\varphi\right)\right)\right|$$

vanishes faster than any power of n^{-1} uniformly in $\varphi_n \leq |\varphi| \leq \pi$. Thus the claim is proved for $b\left(n\right) = 1$.

Recall that

$$-\Re g_n\left(\varphi\right) \geq \frac{1}{\alpha\left(n\right)}\sum_{j=1}^{\alpha(n)} q_{j,n}\left(1-\cos\left(j\varphi\right)\right)x^j.$$

For $b\left(n\right)$ and $\delta, c > 0$ as in condition (3) in Definition 2.1.1, consider

$$\begin{aligned} -\Re g_n\left(\varphi\right) \geq & cr_n\left(\varphi\right) + \frac{1}{\alpha\left(n\right)}\sum_{j=1}^{\alpha(n)}\left(q_{j,n}-c\right)\left(1-\cos\left(j\varphi\right)\right)x^j \\ \geq & cr_n\left(\varphi\right) - \frac{2c}{\alpha\left(n\right)}\sum_{j=1}^{b(n)} x^j \\ \geq & cr_n\left(\varphi\right)\left(1-\frac{2b\left(n\right)}{r_n\left(\varphi\right)\alpha\left(n\right)}x^{b(n)}\right). \end{aligned} \tag{2.1.14}$$

Since $b\left(n\right)/\alpha\left(n\right) \leq 1-\delta$, we have

$$x^{b(n)-\alpha(n)} = \left(x^{\alpha(n)}\right)^{\frac{b(n)-\alpha(n)}{\alpha(n)}} \leq \left(\frac{n}{\alpha\left(n\right)}\right)^{c_1\frac{b(n)-\alpha(n)}{\alpha(n)}} \leq \left(\frac{n}{\alpha\left(n\right)}\right)^{-\delta c_1}$$

by Equation (2.1.12). Because the arguments concerning the properties of $r_n\left(\varphi\right)$ above do not depend on any assumption about $b\left(n\right)$, we may apply Equation (2.1.13) and conclude that the term

$$1-\frac{2b\left(n\right)}{r_n\left(\varphi\right)\alpha\left(n\right)}x^{b(n)}$$

in Equation (2.1.14) converges to 1. This proves the claim. $\square$

2.2. Admissibility: Tools and Example

In Section 2.1 we have implemented the saddle-point method under certain admissibility conditions laid down in Definitions 2.1.1 and 2.1.2. In the present section, we will provide general results which will be applied in the following sections to verify admissibility in specific instances.
The saddle-point asymptotics presented in Lemma 2.2.1 should thus be interpreted in the context of conditions (1) and (2) in Definition 2.1.1.

LEMMA 2.2.1 ([**45**, Lemma 9]). *Given a sequence $(\alpha(n))_{n\in\mathbb{N}}$ such that $2 \leq \alpha(n) \leq n$ for all n, let $x_{n,\alpha}(u)$ be the unique positive solution of*

$$\alpha(n)\, u = \sum_{j=1}^{\alpha(n)} (x_{n,\alpha}(u))^j \tag{2.2.1}$$

for $u > 1$. If $u \geq 3$, there are constants $c, C \in \mathbb{R}$ such that

$$\log(u(\alpha(n) \wedge \log(u))) + c \leq \alpha(n)\log(x_{n,\alpha}(u)) \leq \log(u(\alpha(n) \wedge \log(u))) + C. \tag{2.2.2}$$

Now consider a sequence $(u(n))_{n\in\mathbb{N}}$. If $3 \leq u(n) \leq \mathrm{e}^{\alpha(n)}$ holds for large n, we obtain

$$\alpha(n)\log(x_{n,\alpha}(u(n))) = \log(u(n)\log(u(n))) + \mathcal{O}\left(\frac{\log\log(u(n))}{\alpha(n)\log(u(n))} + \frac{\log(u(n))}{(\alpha(n))^2}\right) \tag{2.2.3}$$

as $n \to \infty$.
Let further $\tilde{\lambda}_{2,n} = \sum_{j=1}^{\alpha(n)} j\,(x_{n,\alpha}(u))^j$ (cf. Equation (2.1.2)). Then, if $u > 1$,

$$\left|\tilde{\lambda}_{2,n} - (\alpha(n))^2 u\right| \leq \frac{(\alpha(n))^2 u}{\log(u)} \tag{2.2.4}$$

holds.

REMARK 2.2.2. The difference in notation between $\tilde{\lambda}_{2,n}$ and $\lambda_{2,n}$ refers to the role played by the cycle weights.

PROOF OF LEMMA 2.2.1. Since $u > 1$, the definition of $x_{n,\alpha}(u)$ leads to $x_{n,\alpha}(u) > 1$ for all u and n. By Equation (2.2.1), we thus have

$$u = \frac{1}{\alpha(n)} \sum_{j=1}^{\alpha(n)} (x_{n,\alpha}(u))^j \leq (x_{n,\alpha}(u))^{\alpha(n)} \leq \alpha(n)\, u. \tag{2.2.5}$$

Since the geometric mean is bounded from above by the arithmetic mean, we obtain

$$(x_{n,\alpha}(u))^{\frac{\alpha(n)+1}{2}} = \left(\prod_{j=1}^{\alpha(n)} (x_{n,\alpha}(u))^j\right)^{\frac{1}{\alpha(n)}} \leq \frac{1}{\alpha(n)} \sum_{j=1}^{\alpha(n)} (x_{n,\alpha}(u))^j = u. \tag{2.2.6}$$

So,

$$u^{\frac{1}{\alpha(n)}} \leq x_{n,\alpha}(u) \leq u^{\frac{2}{\alpha(n)+1}} \tag{2.2.7}$$

follows. The formula for the geometric series applied to Equation (2.2.1) yields

$$\alpha(n)\, u = \sum_{j=1}^{\alpha(n)} (x_{n,\alpha}(u))^j = \frac{(x_{n,\alpha}(u))^{\alpha(n)+1} - x_{n,\alpha}(u)}{x_{n,\alpha}(u) - 1} \tag{2.2.8}$$

or, equivalently,

$$(x_{n,\alpha}(u))^{\alpha(n)} = 1 + \alpha(n)\, u \left(1 - \frac{1}{x_{n,\alpha}(u)}\right). \tag{2.2.9}$$

Let first $\alpha(n) \geq \log(u)$. We have

$$\begin{aligned}
(x_{n,\alpha}(u))^{\alpha(n)} &> \alpha(n)\, u \left(1 - \frac{1}{x_{n,\alpha}(u)}\right) \\
&= \alpha(n)\, u\,(1 - \exp(-\log(x_{n,\alpha}(u)))) \\
&\geq \alpha(n)\, u \left(1 - \exp\left(-\frac{\log(u)}{\alpha(n)}\right)\right) \qquad (2.2.10)
\end{aligned}$$

by Equations (2.2.9) and (2.2.7). Then, since $1-\mathrm{e}^{-t} \geq t\mathrm{e}^{-t}$ for $t \geq 0$ by the Mean Value Theorem,

$$
\begin{aligned}
\left(x_{n,\alpha}\left(u\right)\right)^{\alpha(n)} \geq & u \log\left(u\right) \exp\left(-\frac{\log\left(u\right)}{\alpha\left(n\right)}\right) \\
\geq & \mathrm{e}^{-1} u \log\left(u\right)
\end{aligned}
\tag{2.2.11}
$$

follows, where the last line applies $\log\left(u\right)/\alpha\left(n\right) \leq 1$. Again, by Equations (2.2.9) and (2.2.7),

$$
\begin{aligned}
\left(x_{n,\alpha}\left(u\right)\right)^{\alpha(n)} \leq & 1 + \alpha\left(n\right) u \left(1 - \exp\left(-\frac{2\log\left(u\right)}{\alpha\left(n\right)+1}\right)\right) \\
\leq & 1 + \alpha\left(n\right) u \left(1 - \exp\left(-\frac{2\log\left(u\right)}{\alpha\left(n\right)}\right)\right).
\end{aligned}
\tag{2.2.12}
$$

By the Mean Value Theorem, we conclude $1-\mathrm{e}^{-t} \leq t$ for $t \geq 0$. Thus,

$$
\left(x_{n,\alpha}\left(u\right)\right)^{\alpha(n)} \leq 1 + 2u\log\left(u\right). \tag{2.2.13}
$$

The logarithm is non-decreasing and $\log\left(1+v\right) \leq 1+\log\left(v\right)$ for $v \geq 1$ according to the Mean Value Theorem. By taking the logarithm, Equations (2.2.11) and (2.2.13) combined therefore entail

$$
\log\left(u\log u\right) - 1 \leq \alpha\left(n\right)\log\left(x_{n,\alpha}\left(u\right)\right) \leq \log\left(u\log\left(u\right)\right) + 1 + \log\left(2\right),
$$

which proves the first part of the claim in Equation (2.2.2).
Let now $\alpha\left(n\right) \leq \log\left(u\right)$. Then Equations (2.2.10) and (2.2.12) still hold, and we obtain

$$
\left(x_{n,\alpha}\left(u\right)\right)^{\alpha(n)} > \alpha\left(n\right) u \left(1 - \exp\left(-\frac{\log\left(u\right)}{\alpha\left(n\right)}\right)\right) \geq \left(1-\mathrm{e}^{-1}\right)\alpha\left(n\right) u
$$

and

$$
\left(x_{n,\alpha}\left(u\right)\right)^{\alpha(n)} \leq 1 + \alpha\left(n\right) u \left(1 - \exp\left(-\frac{2\log\left(u\right)}{\alpha\left(n\right)}\right)\right) \leq 1 + \alpha\left(n\right) u.
$$

Taking the logarithm then yields

$$
\log\left(\alpha\left(n\right)u\right) + \log\left(1-\mathrm{e}^{-1}\right) \leq \alpha\left(n\right)\log\left(x_{n,\alpha}\left(u\right)\right) \leq \log\left(\alpha\left(n\right)u\right) + 1,
$$

and the second part of the claim in Equation (2.2.2) is also proved.
In order to prove Equation (2.2.3), we fix a sequence $\left(u\left(n\right)\right)_n$ with $\log\left(3\right) \leq \log\left(u\left(n\right)\right) \leq \alpha\left(n\right)$ for large n and iterate our approach: By Equations (2.2.9) and (2.2.2), we have

$$
\begin{aligned}
& \alpha\left(n\right)\log\left(x_{n,\alpha}\left(u\left(n\right)\right)\right) \\
= & \log\left[1 + \alpha\left(n\right)u\left(n\right)\left(1 - \frac{1}{x_{n,\alpha}\left(u\left(n\right)\right)}\right)\right] \\
= & \log\left[1 + \alpha\left(n\right)u\left(n\right)\left(1 - \exp\left[-\frac{\log\left(u\left(n\right)\log\left(u\left(n\right)\right)\right) + \mathcal{O}\left(1\right)}{\alpha\left(n\right)}\right]\right)\right] \\
= & \log\left[1 + \alpha\left(n\right)u\left(n\right)\left(1 - \exp\left[-\frac{\log\left(u\left(n\right)\log\left(u\left(n\right)\right)\right)}{\alpha\left(n\right)}\right]\left(1 + \mathcal{O}\left(\frac{1}{\alpha\left(n\right)}\right)\right)\right)\right],
\end{aligned}
$$

where the last line applies Taylor's theorem. Further expanding the exponential yields

$$
\begin{aligned}
& \alpha\left(n\right)\log\left(x_{n,\alpha}\left(u\left(n\right)\right)\right) \\
= & \log\left[1 + u\left(n\right)\log\left(u\left(n\right)\log\left(u\left(n\right)\right)\right)\left(1 + \mathcal{O}\left(\frac{\log\left(u\left(n\right)\log\left(u\left(n\right)\right)\right)}{\alpha\left(n\right)}\right)\right) + \mathcal{O}\left(u\left(n\right)\right)\right] \\
= & \log\left[u\left(n\right)\log\left(u\left(n\right)\right)\left(1 + \frac{\log\log\left(u\left(n\right)\right)}{\log\left(u\left(n\right)\right)} + \frac{1}{u\left(n\right)\log\left(u\left(n\right)\right)} + \mathcal{O}\left(\frac{\log\left(u\left(n\right)\right)}{\alpha\left(n\right)} + \frac{1}{\log\left(u\left(n\right)\right)}\right)\right)\right] \\
= & \log\left[u\left(n\right)\log\left(u\left(n\right)\right)\right] + \mathcal{O}\left(\frac{\log\log\left(u\left(n\right)\right)}{\log\left(u\left(n\right)\right)} + \frac{\log\left(u\left(n\right)\right)}{\alpha\left(n\right)}\right),
\end{aligned}
$$

where the error terms are such that the fourth line may apply the the Mean Value Theorem in the given way.

The last step in the proof is showing Equation (2.2.4). Let therefore $u>1$. Since

$$\begin{aligned}\sum_{j=1}^{\alpha(n)} jx^j =& x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{j=1}^{\alpha(n)} x^j\\ =& x\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^{\alpha(n)+1}-x}{x-1}\right]\\ =& \frac{\left(\alpha\left(n\right)+1\right)x^{\alpha(n)+1}-x}{x-1}-\frac{x^{\alpha(n)+2}-x^2}{\left(x-1\right)^2}\end{aligned}$$

holds for real numbers x, substituting $x_{n,\alpha}\left(u\right)$ for x yields

$$\begin{aligned}\sum_{j=1}^{\alpha(n)} j\left(x_{n,\alpha}\left(u\right)\right)^j &= \left(\alpha\left(n\right)+1\right)\alpha\left(n\right)u+\frac{x_{n,\alpha}\left(u\right)}{x_{n,\alpha}\left(u\right)-1}\alpha\left(n\right)-\frac{x_{n,\alpha}\left(u\right)}{x_{n,\alpha}\left(u\right)-1}\alpha\left(n\right)u\\ &= \left(\alpha\left(n\right)\right)^2u+\frac{x_{n,\alpha}\left(u\right)}{x_{n,\alpha}\left(u\right)-1}\alpha\left(n\right)+\alpha\left(n\right)u\left(1-\frac{x_{n,\alpha}\left(u\right)}{x_{n,\alpha}\left(u\right)-1}\right)\\ &= \left(\alpha\left(n\right)\right)^2u-\frac{\alpha\left(n\right)\left(u-x_{n,\alpha}\left(u\right)\right)}{x_{n,\alpha}\left(u\right)-1}\end{aligned}$$

by Equation (2.2.8). We have

$$\alpha\left(n\right)\left(x_{n,\alpha}\left(u\right)-1\right)\geq\alpha\left(n\right)\log\left(x_{n,\alpha}\left(u\right)\right)\geq\log\left(u\right)$$

by the Mean Value Theorem and Equation (2.2.7). Hence, by Equation (2.2.7),

$$\left|\frac{\alpha\left(n\right)\left(u-x_{n,\alpha}\left(u\right)\right)}{x_{n,\alpha}\left(u\right)-1}\right|\leq\left|\frac{\alpha\left(n\right)u}{x_{n,\alpha}\left(u\right)-1}\right|=\left|\frac{\left(\alpha\left(n\right)\right)^2u}{\alpha\left(n\right)\left(x_{n,\alpha}\left(u\right)-1\right)}\right|\leq\frac{\left(\alpha\left(n\right)\right)^2u}{\log\left(u\right)},$$

and the claim is proved. □

The paradigm of an admissible triangular array, which will be of vital importance in the study of random permutations without macroscopic cycles, is the content of the next definition.

DEFINITION 2.2.3. Let α be a sequence and let $\boldsymbol{q}_\vartheta$ denote the triangular array which is given by $q_{j,n}=\vartheta$ for $1\leq j\leq\alpha\left(n\right)$ for some $\vartheta>0$. Write also $x_{n,\vartheta}$ as a shorthand for $x_{n,\boldsymbol{q}_\vartheta}$.

LEMMA 2.2.4. *For $\alpha\left(n\right)$ satisfying Equation (2.0.2), the array $\boldsymbol{q}_\vartheta$ is admissible.*

REMARK 2.2.5. The question whether the array $\boldsymbol{q}_\vartheta$ is still admissible (potentially in a broadened sense) if we consider more general sequences α will be discussed in Section 3.2.1.

PROOF OF LEMMA 2.2.4. We have to verify the conditions in Definition 2.1.1. Note that

$$x_{n,\vartheta}=x_{n,\alpha}\left(\frac{n}{\alpha\left(n\right)\vartheta}\right)\tag{2.2.14}$$

according to the definitions in Equations (2.1.1) and (2.2.1). Due to Equation (2.0.2), $\alpha\left(n\right)\geq\log\left(n/\left(\alpha\left(n\right)\vartheta\right)\right)$ for large n. Condition (1) thus follows from Equation (2.2.14) and Lemma 2.2.1. The second condition is a consequence of $\lambda_{2,n}=\vartheta\tilde{\lambda}_{2,n}$ and Equation (2.2.4). The definition of $\boldsymbol{q}_\vartheta$ entails directly that condition (3) is satisfied with $b\left(n\right)=1$. □

2.3. The Expected Number of Cycles of a Given Length

The following section provides the first application of the saddle-point method developped in Section 2.1 to random permutations without macroscopic cycles. In it we determine the asymptotic behaviour of $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{m(n)}\right]$ for sequences $(m(n))_{n\in\mathbb{N}}$ such that $1 \leq m(n) \leq \alpha(n)$ for all n. Since the asymptotic expected value of $C_{m(n)}$ is sufficient to decide on the form of limit distributions of $C_{m(n)}$, which will be established in Section 2.4, $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{m(n)}\right]$ is an object of great interest. Section 2.3.2 therefore deals with the influence of the sequence α on the asymptotics of expected cycle counts and identifies three classes of α based on the behaviour of the expected number of cycles of maximal length, $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{\alpha(n)}\right]$.
The results of this section have been developped in **[10]** for the case $\vartheta = 1$.
Let α satisfy Equation (2.0.2) throughout this section.

2.3.1. Asymptotics of Expected Cycle Counts. We start with a lemma concerning the normalizing constant of the constrained Ewens measure.

LEMMA 2.3.1. *Fix $\vartheta > 0$. Then the normalizing constant of the conditioned Ewens measure satisfies*

$$Z_{n,\alpha,\vartheta} \sim \frac{\exp\left(\lambda_{0,n}\right)}{x_{n,\vartheta}^{n}\sqrt{2\pi\lambda_{2,n}}} \tag{2.3.1}$$

as $n \to \infty$. Moreover, we have

$$\alpha(n)\log\left(x_{n,\vartheta}\right) = \log\left(\frac{n}{\alpha(n)\vartheta}\log\left(\frac{n}{\alpha(n)\vartheta}\right)\right) + \mathcal{O}\left(\frac{\log\log(n)}{\alpha(n)\log(n)}\right) \tag{2.3.2}$$

and

$$\lambda_{2,n} \sim n\alpha(n) \tag{2.3.3}$$

as $n \to \infty$. In particular,

$$\begin{aligned} x_{n,\vartheta} &> 1, \\ \lim_{n\to\infty} x_{n,\vartheta} &= 1, \end{aligned}$$

and

$$x_{n,\vartheta}^{\alpha(n)} \sim \frac{1}{\vartheta}\frac{n}{\alpha(n)}\log\left(\frac{n}{\alpha(n)}\right)$$

as n tends to infinity.

PROOF. Recall that $Z_{n,\alpha,\vartheta}$ is defined in Equation (2.0.1). By Equation (1.3.6), we have

$$Z_{n,\alpha,\vartheta} = \frac{1}{n!}\sum_{\sigma\in S_{n,\alpha}}\prod_{j=1}^{\alpha(n)}\vartheta^{C_j(\sigma)} = [z^n]\exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta z^j}{j}\right).$$

Since the array $\boldsymbol{q}_\vartheta$ is admissible by Lemma 2.2.4, we may apply Proposition 2.1.4 with $f_n = 1$ (which is trivially admissible). Hence, Equation (2.3.1) is proved. Recall Equation (2.2.14) which states that $x_{n,\vartheta} = x_{n,\alpha}\left(\frac{n}{\alpha(n)\vartheta}\right)$. Equation (2.3.2) thus follows from Equations (2.2.3) and (2.0.2). Also, due to $\lambda_{2,n} = \vartheta\tilde{\lambda}_{2,n}$, we have

$$\lambda_{2,n} \sim \vartheta\frac{n\alpha(n)}{\vartheta} = n\alpha(n)$$

by Equation (2.2.4), which proves Equation (2.3.3). □

Define

$$\mu_{m(n)}(n) := \mu_{m(n)}^{(\vartheta)}(n) := \vartheta\frac{x_{n,\vartheta}^{m(n)}}{m(n)} \tag{2.3.4}$$

for sequences $(m(n))_{n\in\mathbb{N}}$ and $\vartheta > 0$.

PROPOSITION 2.3.2 ([**10**, Proposition 2.1]). *Fix $\vartheta > 0$. Then for all sequences $(m(n))_{n\in\mathbb{N}}$ with values in $\mathbb{N}$ such that $m(n) \le \alpha(n)$ for all $n \in \mathbb{N}$, we have*

$$\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{m(n)}\right] \sim \mu_{m(n)}(n)$$

as $n \to \infty$.

PROOF. By applying Equation (1.3.6) with $q_{m(n),n} = \vartheta \mathrm{e}^s$ for $s \in \mathbb{R}$ and $q_{j,n} = \vartheta$ otherwise, we obtain

$$\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\mathrm{e}^{sC_{m(n)}}\right] = \frac{1}{Z_{n,\alpha,\vartheta}}[z^n]\exp\left(\vartheta\left(\mathrm{e}^s - 1\right)\frac{z^{m(n)}}{m(n)}\right)\exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta z^j}{j}\right).$$

Differentiation with respect to s yields

$$\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{m(n)}\right] = \frac{\vartheta}{Z_{n,\alpha,\vartheta}}[z^n]\frac{z^{m(n)}}{m(n)}\exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta z^j}{j}\right)$$

when evaluated at $s = 0$. By considering the array $\boldsymbol{q}_\vartheta$ and defining

$$f_n(z) := \frac{z^{m(n)}}{m(n)},$$

we are in the situation of Proposition 2.1.4 if we can verify admissibility. The array $\boldsymbol{q}_\vartheta$ is admissible due to Lemma 2.2.4. Since f_n is entire for all n, condition (1) in Definition 2.1.2 holds. Condition (2) follows from

$$|f_n(z)| \le \frac{|z|^{m(n)}}{m(n)} = |f_n(x_{n,\vartheta})|$$

for $z \in \partial B_{x_{n,\vartheta}}(0)$. Since $f_n'(z) = z^{m(n)-1}$, we have

$$|f_n'(z)| \le |f_n(x_{n,\vartheta})|\frac{m(n)}{x_{n,\vartheta}}$$

for all $z \in \partial B_{x_{n,\vartheta}}(0)$. Condition (3) is thus a consequence of $m(n) \le \alpha(n) = o\left(n^{5/12}(\alpha(n))^{7/12}\right)$ and Lemma 2.3.1. By Proposition 2.1.4, we arrive at

$$\begin{aligned}\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{m(n)}\right] &\sim \frac{\vartheta}{Z_{n,\alpha,\vartheta}}\frac{x_{n,\vartheta}^{m(n)}}{m(n)}\frac{\exp(\lambda_{0,n})}{x_{n,\vartheta}^n\sqrt{2\pi\lambda_{2,n}}}\\ &\sim \vartheta\frac{x_{n,\vartheta}^{m(n)}}{m(n)}\\ &= \mu_{m(n)}(n),\end{aligned}$$

where the second line applies Lemma 2.3.1. □

REMARK 2.3.3. We could also treat higher moments of $C_{m(n)}$ with the machinery of the proof of Proposition 2.3.2, but Section 2.4 will show that the limit distributions of the $C_{m(n)}$ only depend on the respective asymptotic expected values, so we concentrate on the first moment.

2.3.2. Special Cases and the Role of the Maximal Cycle Length. In this section we will present the different regimes for the expected values of individual cycle numbers. We focus on the special cases when the maximal cycle length is given by $\alpha(n) := n^\beta$ for $\beta \in (0,1)$ or $\alpha(n) := \sqrt{n\log(n)}$, but the main results generalize. It will emerge that the picture strongly depends on the specific value of β. If $\beta \le \frac{1}{2}$, there will be a regime of diverging asymptotic expectation of the numbers of long cycles. For $\beta > \frac{1}{2}$, no such regime occurs and the individual numbers of long cycles even converge to 0. If $\alpha(n) := \sqrt{n\log(n)}$, there is no regime of diverging expectation, but the numbers of long cycles do not converge to 0. An important point of reference is the classical model of random permutations under the Ewens measure in which the expected value of the cycle number $C_{m(n)}$ with $m(n) = o(n)$ exhibits the behaviour $\mathbb{E}_n^{(\vartheta)}\left[C_{m(n)}\right] \sim \frac{\vartheta}{m(n)}$ (cf. Section 1.1.2.2). Hence, the change of behaviour depending on $\alpha(n)$ may be interpreted in the way that, as β increases, the model of constrained permutations more and more resembles the classical model in which the expectation of counts of long cycles vanishes asymptotically.

We first consider the case $\alpha(n) = n^{\beta}$. As was shown in Proposition 2.3.2, the asymptotic expected value of an individual cycle number $C_{m(n)}$ in the model of constrained permutations (i.e., under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$) is given by $\mu_{m(n)}(n) = \vartheta \frac{x_{n,\vartheta}^{m(n)}}{m(n)}$. We say that the asymptotics are classical if $x_{n,\vartheta}^{m(n)} \to 1$ holds (note that $x_{n,\vartheta} \to 1$ according to Lemma 2.3.1) since $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{m(n)}\right] \sim \mathbb{E}_{n}^{(\vartheta)}\left[C_{m(n)}\right] \sim \vartheta/m(n)$ as $n \to \infty$ in this case. The asymptotic behaviour of $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{m(n)}\right]$ thus does not reflect any influence of the constraint in this regime. By Equation (2.3.2), $(m(n))_{n\in\mathbb{N}}$ being classical is equivalent to

$$\begin{aligned}
\log\left(x_{n,\vartheta}^{m(n)}\right) =& \frac{m(n)}{n^{\beta}} n^{\beta} \log(x_{n,\vartheta}) \\
=& \frac{m(n)}{n^{\beta}} \left[\log\left(\frac{n^{1-\beta}}{\vartheta} \log\left(\frac{n^{1-\beta}}{\vartheta}\right)\right) + \mathcal{O}\left(\frac{\log(\log(n))}{\log(n)}\right)\right] \\
\to& 0
\end{aligned} \tag{2.3.5}$$

as $n \to \infty$. The cycle number $C_{m(n)}$ therefore belongs to the classical regime if

$$m(n) = o\left(\frac{n^{\beta}}{\log(n)}\right). \tag{2.3.6}$$

If $m(n)$ stays bounded, $\liminf_{n\to\infty} \mu_{m(n)}(n) > 0$. Cycle numbers in the classical regime with $m(n) \to \infty$ satisfy $\lim_{n\to\infty} \mu_{m(n)}(n) = 0$. Section 2.5 will supplement the previous results by establishing simultaneous convergence in total variation distance of all cycle numbers of lengths $o\left(\frac{n^{\beta}}{\log(n)}\right)$ to independent Poisson-distributed random variables, thus replicating the classical convergence result. Hence, in the limit of large n, short cycles in the precise sense of Equation (2.3.6) are not influenced by imposing the constraint that n^{β} be the maximal cycle length.
The second regime is characterized by $\liminf_{n\to\infty} x_{n,\vartheta}^{m(n)} > 1$ and $\mu_{m(n)}(n) \to 0$. Intuitively, such a cycle number still converges to 0 as it does in the classical case, but the asymptotic behaviour of the expected value now deviates slightly. From a different point of view, this means that the expected number of indices in cycles of length $m(n)$, $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[m(n) C_{m(n)}\right] \sim \vartheta x_{n,\vartheta}^{m(n)}$, starts to rise due to the constraint. The two conditions translate into

$$m(n) = \Omega\left(\frac{n^{\beta}}{\log(n)}\right), \tag{2.3.7}$$

i.e. there are constants $C, N > 0$ such that $m(n) \geq C\frac{n^{\beta}}{\log(n)}$ for all $n \geq N$, by Equation (2.3.5) and

$$\limsup_{n\to\infty} \left(\log\left(x_{n,\vartheta}^{m(n)}\right) - \log(m(n))\right) = -\infty.$$

There are always cycle numbers in this regime since, e.g., the choice $m(n) = \left\lfloor \frac{n^{\beta}}{\log(n)} \right\rfloor$ fulfils both conditions. Now fix n and consider the function $m \mapsto \vartheta \frac{x_{n,\vartheta}^{m}}{m}$. Its derivative is given by

$$\frac{\partial}{\partial m} \vartheta \frac{x_{n,\vartheta}^{m}}{m} = \vartheta \frac{m \log(x_{n,\vartheta}) - 1}{m^2} x_{n,\vartheta}^{m}.$$

Consider the root $m_{min}(n) = 1/\log(x_{n,\vartheta})$. Then, $m \mapsto \vartheta \frac{x_{n,\vartheta}^{m}}{m}$ is decreasing for $m \leq m_{min}(n)$ and increasing for $m \geq m_{min}(n)$. So

$$m_{min}(n) \sim \frac{1}{1-\beta} \frac{n^{\beta}}{\log(n)}$$

(by Equation (2.3.2)) marks the position of the minimum with

$$\mu_{\lfloor m_{min}(n) \rfloor}(n) \sim \mathrm{e}\vartheta(1-\beta) \frac{\log(n)}{n^{\beta}}.$$

Note that the choice $m(n) := \lfloor m_{min}(n) \rfloor$ belongs to the second regime. From this point on, the asymptotic expected cycle count increases with the cycle length.
The paradigm of the third regime is convergence of the asymptotic expected value of a cycle count to a positive real number for a cycle number which would converge to 0 in the classical case. We define it in a slightly more general way by positing $m(n) \to \infty$ and $0 < \liminf_{n\to\infty} \mu_{m(n)}(n) \leq$

$\limsup_{n\to\infty} \mu_{m(n)}(n) < \infty$. Indeed, as the cycles need to be longer than in the second regime, Equation (2.3.7) also has to be satisfied. By applying Equation (2.3.5), we obtain

$$\begin{aligned}
\mu_{m(n)}(n) =& \vartheta \frac{x_{n,\vartheta}^{m(n)}}{m(n)} \\
=& \vartheta \frac{\exp\left[\frac{m(n)}{n^\beta}\left[\log\left(\frac{n^{1-\beta}}{\vartheta}\log\left(\frac{n^{1-\beta}}{\vartheta}\right)\right) + \mathcal{O}\left(\frac{\log(\log(n))}{\log(n)}\right)\right]\right]}{m(n)} \\
\sim& \vartheta^{1-m(n)/n^\beta} \frac{n^{(1-\beta)\frac{m(n)}{n^\beta}}}{m(n)} \cdot \left(\log\left(\frac{n^{1-\beta}}{\vartheta}\right)\right)^{\frac{m(n)}{n^\beta}}.
\end{aligned} \tag{2.3.8}$$

At this point, the precise value of β is of paramount importance. One easily sees that, by Equation (2.3.7), $\mu_{m(n)}(n) \to 0$ if $\beta > \frac{1}{2}$, so there may never be convergence to a positive real number in this case. The third regime may therefore only occur if $\beta \le \frac{1}{2}$. For such β, consider

$$\log\left(\mu_{m(n)}(n)\right) = \log(\vartheta) + \frac{m(n)}{n^\beta}\left[\log\left(\frac{n^{1-\beta}}{\vartheta}\log\left(\frac{n^{1-\beta}}{\vartheta}\right)\right) + \mathcal{O}\left(\frac{\log(\log(n))}{\log(n)}\right)\right] - \log(m(n)).$$

Note that $\frac{\beta}{1-\beta} \in (0,1]$ if $0 < \beta \le \frac{1}{2}$. For $v \in \mathbb{R}$ and

$$c_v(n) := \frac{\beta}{1-\beta} - \frac{\beta}{(1-\beta)^2}\frac{\log\left(\log\left(n^{1-\beta}/\vartheta\right)\right)}{\log(n)} + \frac{v}{1-\beta}\frac{1}{\log(n)},$$

we have that $m(n) := \lfloor c_v(n) n^\beta \rfloor \le n^\beta$ for large n and

$$\begin{aligned}
\log\left(\mu_{\lfloor c_v(n)n^\beta\rfloor}(n)\right) =& \log(\vartheta) + c_v(n)\left[\log\left(\frac{n^{1-\beta}}{\vartheta}\right) + \log\log\left(\frac{n^{1-\beta}}{\vartheta}\right) + o(1)\right] \\
&- \log(c_v(n)) - \beta\log(n) + o(1) \\
=& (1-c_v(n))\log(\vartheta) - \log(c_v(n)) - \frac{\beta}{1-\beta}\log\log\left(\frac{n^{1-\beta}}{\vartheta}\right) \\
&+ v + \frac{\beta}{1-\beta}\log\log\left(\frac{n^{1-\beta}}{\vartheta}\right) + o(1) \\
\to& \frac{1-2\beta}{1-\beta}\log(\vartheta) + v - \log\left(\frac{\beta}{1-\beta}\right)
\end{aligned}$$

as $n \to \infty$. So

$$\lim_{n\to\infty} \mu_{\lfloor c_v(n)n^\beta\rfloor}(n) = \frac{1-\beta}{\beta}\vartheta^{\frac{1-2\beta}{1-\beta}}\mathrm{e}^v,$$

which entails that, if $\beta \le 1/2$, then for any non-negative real number, there is a sequence of cycle lengths such that the expected cycle numbers converge to that number. The fact that the third regime is very narrow can be glimpsed from

$$\mu_{\lfloor cn^\beta\rfloor}(n) \to 0$$

for $c < \frac{\beta}{1-\beta}$ and

$$\mu_{\lfloor cn^\beta\rfloor}(n) \to \infty$$

for $c \ge \frac{\beta}{1-\beta}$ by Equation (2.3.8). It requires modifying the critical value of $\frac{\beta}{1-\beta}$ by a term of order $o(1)$ (see the definition of c_v) to identify the corresponding scale.

We have thus already seen that, if $\beta \le \frac{1}{2}$, there exists a fourth regime of cycle numbers which exhibit diverging asymptotic expected values. In this case, the largest possible expected values are obtained by considering the number of cycles of maximal length: If $m(n) := n^\beta$, then

$$\mu_{n^\beta}(n) \sim (1-\beta)n^{1-2\beta}\log(n)$$

by Equation (2.3.8), which does not depend on the value of ϑ.

It is now an interesting question to ask whether there is a sequence α such that the model has cycle numbers in the first three regimes, but no cycle numbers with diverging expected values. The choice $\alpha(n) = \sqrt{n\log(n)}$ has these properties. Since the general results above may be extended to this case, it suffices to consider the number of cycles of maximal length $m(n) = \alpha(n) = \sqrt{n \log n}$. We conclude

$$\mu_{\alpha(n)}(n) = \frac{x_{n,\vartheta}^{\alpha(n)}}{\alpha(n)} \to \frac{1}{2}.$$

So we see that there are three classes of possible sequences α for which the cycle numbers exhibit qualitatively different behaviour. That $\beta = \frac{1}{2}$ marks the threshold can be understood as follows: In all cases, n indices have to be distributed among cycles of lengths up to n^β, so the total number of cycles has to exceed $n^{1-\beta}$. The ratio of both sequences $n^\beta / n^{1-\beta} = n^{1-2\beta}$ diverges for $\beta < \frac{1}{2}$, so there have to be cycle lengths with diverging cycle counts in this case. If, on the other hand, $\beta > \frac{1}{2}$, we have $n^{1-2\beta} \to 0$, which means that there are many possible lengths to allocate the cycles to. It is therefore possible that all cycle counts stay bounded.
Figures 2.3.1, 2.3.2, and 2.3.3 illustrate the different regimes for three special cases satisfying $\beta \leq \frac{1}{2}$, $\alpha(n) = \sqrt{n \log(n)}$, and $\beta > \frac{1}{2}$, respectively. Emphasis should be put on the values of $\mu_{\alpha(n)}(n)$ attained by the longest cycles in each instance.

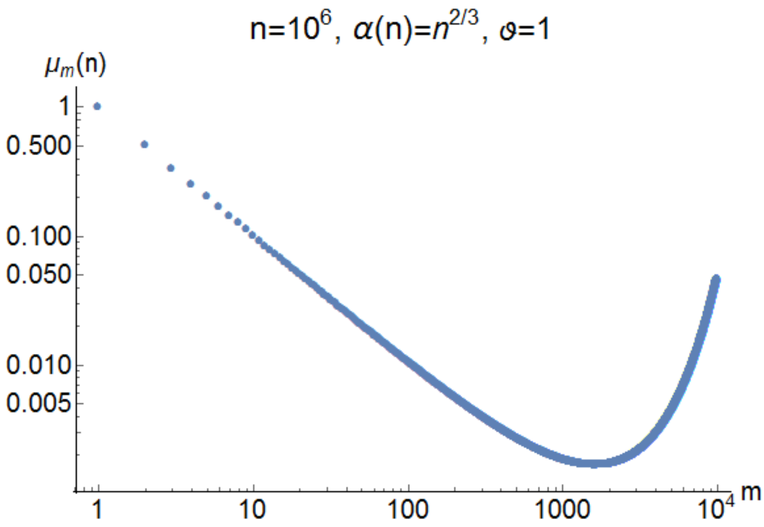

FIGURE 2.3.1. Asymptotic expected values of C_m under $\mathbb{P}_{n,\alpha}^{(1)}$ with $\alpha(n) = n^{2/3}$ in doubly logarithmic scale

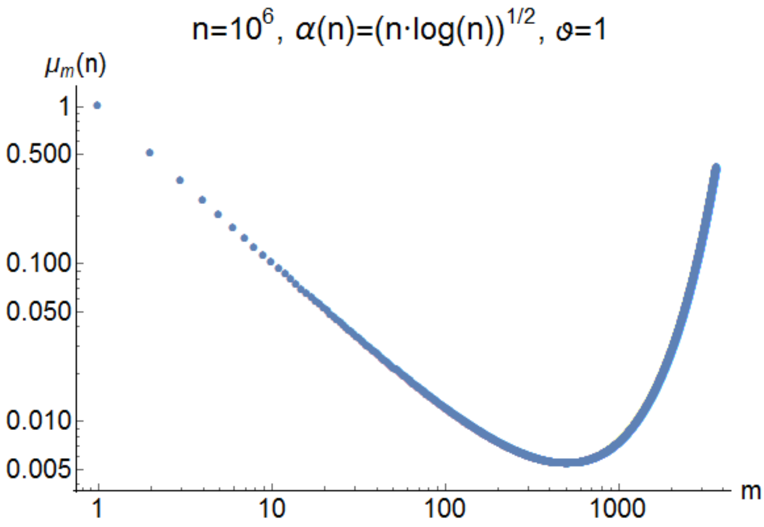

FIGURE 2.3.2. Asymptotic expected values of C_m under $\mathbb{P}_{n,\alpha}^{(1)}$ with $\alpha(n) = \sqrt{n \log(n)}$ in doubly logarithmic scale

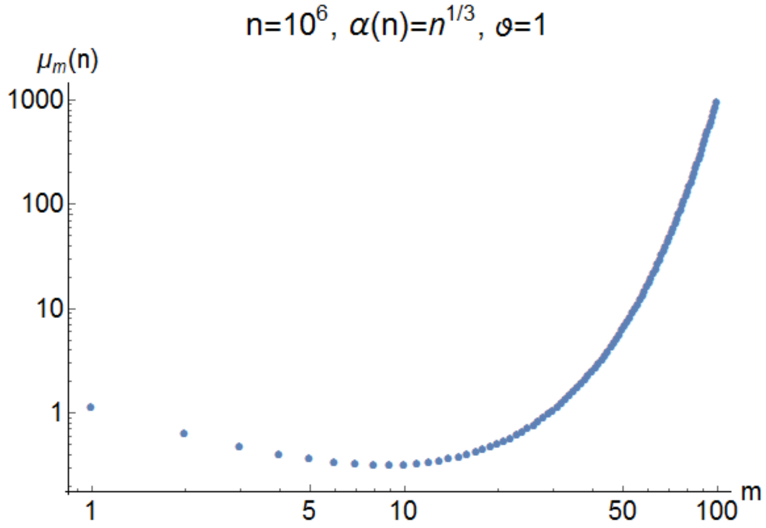

FIGURE 2.3.3. Asymptotic expected values of C_m under $\mathbb{P}^{(1)}_{n,\alpha}$ with $\alpha(n) = n^{1/3}$ in doubly logarithmic scale

One should further note that the dependence of the expected cycle numbers on ϑ is most marked for short cycles. Indeed, by Equation (2.3.8), the influence of ϑ steadily diminishes with rising $m(n)$ in leading order and vanishes completely for lengths which are asymptotically equivalent to $\alpha(n)$. This phenomenon will also be recovered in more subtle situations (see, in particular, Section 2.7). Intuitively, the importance of the constraint grows with the cycle length under consideration and in the end completely overrides the bias incurred by ϑ. Furthermore, the borderlines between the four regimes defined above and the role of $\beta = \frac{1}{2}$ do not depend on ϑ.
Having investigated the expected values, the following Section 2.4 deals with the (joint) distributions of cycle numbers in all four regimes.

2.4. Limit Distributions of Individual Cycle Numbers

In this section we consider the joint distributions of individual cycle numbers, thereby building on our results about asymptotic expected values in Section 2.3.2. We present three propositions which hold for cycle numbers in different regimes. If the expectation of the cycle numbers stays bounded (regimes 1, 2, and 3), there are two results. On the one hand, for converging asymptotic expected values, Proposition 2.4.1 establishes weak convergence to (possibly trivial) Poisson-distributed random variables. On the other hand, we can employ a technique called tilting, which is known from large deviation theory, to obtain convergence of the tilted cycle counts to non-trivial Poisson-distributed random variables. This approach is implemented in Proposition 2.4.4 and enables us to extract information when the untilted cycle numbers converge to 0. It further applies to sequences with slowly diverging expected cycle counts. For general cycle numbers with diverging expected value (regime 4), Proposition 2.4.8 provides a central limit theorem for the centred and rescaled random variables under a certain additional condition (which we believe to be of mere technical nature). In all three propositions, we have independence of the different cycle numbers in the limit. The content of Sections 2.4.2 and 2.4.3 has been developped in **[10]** for $\vartheta = 1$, the assumptions in Section 2.4.2 have, however, been weakened considerably.
We assume in the following that α satisfies Equation (2.0.2).

2.4.1. Converging Expected Values.

In this section we consider the distributions of $C_{m(n)}$ for sequences of cycle lengths $(m(n))_{n\in\mathbb{N}}$ such that the corresponding asymptotic expected values $\mu_{m(n)}(n)$ converge to a real number μ. The cycle lengths under consideration therefore belong to regime 1, 2 or 3.
The result presented in Proposition 2.4.1 has been stated in the second remark after Theorem 2.3 in **[10]**.

Proposition 2.4.1. *Let $(m_k(n))_{n\in\mathbb{N}}$, $k = 1, \ldots, K$, be sequences satisfying $m_k(n) \leq \alpha(n)$ and $m_{k_1}(n) \neq m_{k_2}(n)$ if $k_1 \neq k_2$ for large n such that*

$$\mu_{m_k(n)}(n) \to \mu_k \in [0, \infty)$$

for all k. Then,

$$\left(C_{m_1(n)}, \ldots, C_{m_K(n)}\right) \xrightarrow{d} (Z_1, \ldots, Z_K)$$

as $n \to \infty$. Here, for each n, $\left(C_{m_1(n)}, \ldots, C_{m_K(n)}\right)$ is considered under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$, and the Z_k are independent and Poisson-distributed random variables with parameters $\mathbb{E}[Z_k] = \mu_k$.

Proof. By Equation (1.3.6), we have

$$\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\prod_{k=1}^{K} \mathrm{e}^{s_k C_{m_k(n)}}\right] = \frac{1}{Z_{n,\alpha,\vartheta}}\left[z^n\right] \exp\left[\sum_{k=1}^{K}\left(\mathrm{e}^{s_k}-1\right)\frac{\vartheta z^{m_k(n)}}{m_k(n)} + \sum_{j=1}^{\alpha(n)}\frac{\vartheta z^j}{j}\right] \tag{2.4.1}$$

and want to apply Proposition 2.1.4. Fix $s_k \geq 0$ for all k and consider the functions

$$f_n(z) = \exp\left[\sum_{k=1}^{K}\left(\mathrm{e}^{s_k}-1\right)\frac{\vartheta z^{m_k(n)}}{m_k(n)}\right].$$

The relevant array is $\boldsymbol{q}_\vartheta$, which is admissible by Lemma 2.2.4. So we only have to check admissibility of the perturbations (see Definition 2.1.2). The first condition holds since the functions f_n are entire. Condition (2) is a consequence of

$$|f_n(z)| \leq |f_n(x_{n,\vartheta})|$$

for $|z| = x_{n,\vartheta}$, which follows from $s_k \geq 0$. Because of

$$f_n'(z) = \sum_{k=1}^{K}\left(\mathrm{e}^{s_k}-1\right)\vartheta z^{m_k(n)-1} f_n(z),$$

we conclude

$$\begin{aligned}\| f_n \|_n \leq & n^{-\frac{5}{12}} (\alpha(n))^{-\frac{7}{12}} \sum_{k=1}^{K} \left(\mathrm{e}^{s_k} - 1\right) \vartheta x_{n,\vartheta}^{m_k(n)-1} \\ = & n^{-\frac{5}{12}} (\alpha(n))^{-\frac{7}{12}} \sum_{k=1}^{K} \left(\mathrm{e}^{s_k} - 1\right) m_k(n) \frac{\mu_{m_k(n)}(n)}{x_{n,\vartheta}} \\ \leq & n^{-\frac{5}{12}} \alpha(n)^{\frac{5}{12}} \sum_{k=1}^{K} \left(\mathrm{e}^{s_k} - 1\right) \frac{\mu_{m_k(n)}(n)}{x_{n,\vartheta}}.\end{aligned}$$

Since the $\mu_{m_k(n)}(n)$ are convergent and $x_{n,\vartheta} \to 1$,

$$\| f_n \|_n \xrightarrow{n\to\infty} 0$$

follows. So we have verified the third condition. Proposition 2.1.4 thus yields

$$\mathbb{E}_{n,\alpha}^{(\vartheta)} \left[\prod_{k=1}^{K} \mathrm{e}^{s_k C_{m_k(n)}} \right] \sim f_n(x_{n,\vartheta})$$

by Lemma 2.3.1. From

$$\lim_{n\to\infty} \frac{\vartheta x_{n,\vartheta}^{m_k(n)}}{m_k(n)} = \mu_k$$

we conclude

$$f_n(x_{n,\vartheta}) \xrightarrow{n\to\infty} \exp\left[\sum_{k=1}^{K} \left(\mathrm{e}^{s_k} - 1\right) \mu_k \right],$$

and, by Corollary 1.2.7, the claim is proved. □

Proposition 2.4.1 immediately entails the following

Corollary 2.4.2. *For $K \in \mathbb{N}$, we have*

$$(C_1, \ldots, C_K) \xrightarrow{d} (Z_1, \ldots, Z_K),$$

where the Z_k are independent Poisson-distributed random variables with parameters $\mathbb{E}[Z_k] = \frac{\vartheta}{k}$.

Remark 2.4.3. Corollary 2.4.2 shows that, in the limit, the finite-dimensional distributions of counts of short cycles are not affected by imposing the constraint of a maximal cycle length. This result will be significantly strengthened in Section 2.5.

2.4.2. Tilting. If we consider a sequence of cycle lengths $(m(n))_{n\in\mathbb{N}}$ with $\mu_{m(n)}(n) \to 0$, then Proposition 2.4.1 does not provide us with much information about the distribution of $C_{m(n)}$ since the limit is trivial. Additionally, the proposition even fails to hold if $\mu_{m(n)}(n)$ stays bounded, but does not converge to a fixed number. The following Proposition 2.4.4 deals with both cases and further applies to sequences with slowly diverging expected cycle numbers, and it does so by suitably tilting the cycle counts in question, which is motivated by a technique well-known in the theory of large deviations (cf., e.g., [**24**, Section II.7] or [**19**, Section III.4]). In particular, Proposition 2.4.4 thus offers a unified approach to all cycle counts in the regimes 1,2, and 3 (see Section 2.3.2).

For any cycle number C_m and $\nu \in [0, \infty)$, we define the tilted cycle number $C_m^{(\nu)}$ as the $\mathbb{N}_0$-valued random variable (with respect to a new probability measure $\mathbb{P}$) with

$$\mathbb{P}\left[C_m^{(\nu)} = l\right] = \frac{1}{Z^{(n)}} \frac{\mathrm{e}^{\nu}}{\nu^l} \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[C_m = l\right] \tag{2.4.2}$$

for all $l \in \mathbb{N}_0$. Here, $Z^{(n)}$ denotes a normalizing constant. If we consider several random variables at once, a simultaneous tilt is to be defined in an analogous way by

$$\mathbb{P}\left[C_{m_1}^{(\nu_1)} = l_1, \ldots, C_{m_K}^{(\nu_K)} = l_K\right] = \frac{1}{Z^{(n)}} \left(\prod_{k=1}^{K} \frac{\mathrm{e}^{\nu_k}}{\nu_k^l} \right) \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[C_{m_1} = l_1, \ldots, C_{m_K} = l_K\right] \tag{2.4.3}$$

for all $l_1, \ldots, l_K \in \mathbb{N}_0$. The following Proposition 2.4.4 extends the result obtained in [**10**] by also including sequences $(m(n))_{n\in\mathbb{N}}$ of cycle lengths whose asymptotic expected cycle counts diverge slowly.

PROPOSITION 2.4.4 ([**10**, Theorem 2.3]). *Let the sequence α be as in Equation (2.0.2) and let $(m_1(n))_{n\in\mathbb{N}}, \ldots, (m_K(n))_{n\in\mathbb{N}}$ be $\mathbb{N}$-valued sequences with $m_k(n) \leq \alpha(n)$ such that there is $\delta > 0$ with*

$$\sum_{k=1}^{K} \mu_{m_k(n)}(n) \leq \left(\frac{5}{24} - \delta\right) \log\left(\frac{n}{\alpha(n)}\right) \tag{2.4.4}$$

for large n. Then,

$$\left(C_{m_1(n)}^{\left(\mu_{m_1(n)}(n)\right)}, \ldots, C_{m_K(n)}^{\left(\mu_{m_K(n)}(n)\right)}\right) \xrightarrow{d} (Z_1, \ldots, Z_K)$$

holds. Here, for each $n \in \mathbb{N}$, $C_{m_1(n)}^{\left(\mu_{m_1(n)}(n)\right)}, \ldots, C_{m_K(n)}^{\left(\mu_{m_K(n)}(n)\right)}$ are to be considered as arising from tilting the respective cycle counts under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$, and the Z_k are independent Poisson-distributed random variables with parameters $\mathbb{E}[Z_k] = 1$.

REMARK 2.4.5. The assumption in Equation (2.4.4) is in particular fulfilled for sequences of cycle lengths $(m_k(n))_{n\in\mathbb{N}}$ with $\limsup_{n\to\infty} \mu_{m_k(n)}(n) < \infty$ for all k. Since $\alpha(n) \geq n^{a_1}$ by Equation (2.0.2), a sufficient condition for this to hold is $m_k(n) \leq c\alpha(n)$ with $c < \frac{a_1}{1-a_1}$ (see the discussion in Section 2.3.2).

PROOF OF PROPOSITION 2.4.4. Let us write $\mu_{k,n} := \mu_{m_k(n)}(n)$, $C_{m_k} := C_{m_k(n)}$, and $\widetilde{C}_{m_k} := C_{m_k}^{(\mu_{k,n})}$ to ease notation. Fix $s_k \geq 0$. Then, by Equation (2.4.3),

$$\begin{aligned}
\mathbb{E}\left[\prod_{k=1}^{K} \mathrm{e}^{s_k \widetilde{C}_{m_k}}\right] &= \sum_{l_1=0}^{\infty} \cdots \sum_{l_K=0}^{\infty} \left(\prod_{k=1}^{K} \mathrm{e}^{s_k l_k}\right) \mathbb{P}\left[\widetilde{C}_{m_1} = l_1, \ldots, \widetilde{C}_{m_K} = l_K\right] \\
&= \frac{1}{Z^{(n)}} \sum_{l_1=0}^{\infty} \cdots \sum_{l_K=0}^{\infty} \left(\prod_{k=1}^{K} \frac{\mathrm{e}^{s_k l_k + \mu_{k,n}}}{\mu_{k,n}^{l_k}}\right) \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[C_{m_1} = l_1, \ldots, C_{m_K} = l_K\right] \\
&= \frac{\mathrm{e}^{\sum_{k=1}^{K} \mu_{k,n}}}{Z^{(n)}} \sum_{l_1=0}^{\infty} \cdots \sum_{l_K=0}^{\infty} \left(\prod_{k=1}^{K} \mathrm{e}^{l_k(s_k - \log(\mu_{k,n}))}\right) \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[C_{m_1} = l_1, \ldots, C_{m_K} = l_K\right] \\
&= \frac{\mathrm{e}^{\sum_{k=1}^{K} \mu_{k,n}}}{Z^{(n)}} \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\prod_{k=1}^{K} \mathrm{e}^{(s_k - \log(\mu_{k,n}))C_{m_k}}\right]
\end{aligned}$$

holds. So we have established how the moment-generating function of the tilted cycle counts can be obtained from the MGF of their untilted counterparts. Recall that

$$\begin{aligned}
&\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\prod_{k=1}^{K} \mathrm{e}^{(s_k - \log(\mu_{k,n}))C_{m_k}}\right] \\
=& \frac{1}{Z_{n,\alpha,\vartheta}} [z^n] \exp\left(\sum_{k=1}^{K} \left(\mathrm{e}^{s_k - \log(\mu_{k,n})} - 1\right) \frac{\vartheta z^{m_k(n)}}{m_k(n)}\right) \exp\left(\sum_{j=1}^{\alpha(n)} \frac{\vartheta z^j}{j}\right)
\end{aligned}$$

by Equation (2.4.1) and let

$$f_n(z) := \exp\left(\sum_{k=1}^{K} \left(\mathrm{e}^{s_k - \log(\mu_{k,n})} - 1\right) \frac{\vartheta z^{m_k(n)}}{m_k(n)}\right).$$

Our goal is to apply Proposition 2.1.4 with the array $\boldsymbol{q}_\vartheta$. Note again that $\boldsymbol{q}_\vartheta$ is admissible by Lemma 2.2.4 and that the functions f_n are entire, so condition (1) in Definition 2.1.2 holds. We are now going to check conditions (2) and (3). When $s_k \geq \log(\mu_{k,n})$, we have

$$\begin{aligned}\Re\left(\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right)\frac{\vartheta z^{m_k(n)}}{m_k(n)} &\leq \left|\left(\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right)\frac{\vartheta z^{m_k(n)}}{m_k(n)}\right| \\ &= \left(\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right)\frac{\vartheta x_{n,\vartheta}^{m_k(n)}}{m_k(n)}\end{aligned}$$

for $|z| = x_{n,\vartheta}$. When $0 \leq s_k < \log(\mu_{k,n})$, due to the definition of $\mu_{k,n} = \mu_{m_k(n)}(n)$,

$$\left(\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right)\frac{\vartheta x_{n,\vartheta}^{m_k(n)}}{m_k(n)} \geq -\mu_{k,n}$$

holds. Moreover, we have

$$\begin{aligned}\Re\left(\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right)\frac{\vartheta z^{m_k(n)}}{m_k(n)} \leq &\mu_{k,n} \\ &\leq \left(\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right)\frac{\vartheta x_{n,\vartheta}^{m_k(n)}}{m_k(n)} + 2\mu_{k,n}.\end{aligned}$$

From our assumption in Equation (2.4.4),

$$\sum_{k=1}^{K}\mu_{k,n} \leq \left(\frac{5}{24}-\delta\right)\log\left(\frac{n}{\alpha(n)}\right),$$

we deduce for all s_k that

$$\begin{aligned}\Re\sum_{k=1}^{K}\left(\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right)\frac{\vartheta z^{m_k(n)}}{m_k(n)} &\leq \sum_{k=1}^{K}\left(\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right)\frac{\vartheta x_{n,\vartheta}^{m_k(n)}}{m_k(n)} + 2\sum_{k=1}^{K}\mu_{k,n} \\ &\leq \sum_{k=1}^{K}\left(\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right)\frac{\vartheta x_{n,\vartheta}^{m_k(n)}}{m_k(n)} + \left(\frac{5}{12}-2\delta\right)\log\left(\frac{n}{\alpha(n)}\right).\end{aligned}$$

Hence,

$$|f_n(z)| \leq \left(\frac{n}{\alpha(n)}\right)^{\frac{5}{12}-2\delta}|f_n(x_{n,\vartheta})| \tag{2.4.5}$$

for all $|z| = x_{n,\vartheta}$ and large n. This proves condition (2). After a short calculation, we arrive at

$$f_n'(z) = \sum_{k=1}^{K}\left(\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right)\vartheta z^{m_k(n)-1}f_n(z).$$

By Equation (2.4.5), we have

$$\frac{|f_n'(z)|}{|f_n(x_{n,\vartheta})|} \leq \left(\frac{n}{\alpha(n)}\right)^{\frac{5}{12}-2\delta}\sum_{k=1}^{K}\left|\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right|\vartheta x_{n,\vartheta}^{m_k(n)-1}$$

for $|z| = x_{n,\vartheta}$. If, on the one hand, $s_k \geq \log(\mu_{k,n})$, we obtain

$$\left|\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right|\vartheta x_{n,\vartheta}^{m_k(n)-1} \leq \frac{\vartheta \mathrm{e}^{s_k}}{\mu_{k,n}}x_{n,\vartheta}^{m_k(n)-1} \leq \mathrm{e}^{s_k}m_k(n) \leq \mathrm{e}^{s_k}\alpha(n).$$

If, on the other hand, $0 \leq s_k < \log(\mu_{k,n})$, we conclude

$$\left|\mathrm{e}^{s_k-\log(\mu_{k,n})}-1\right|\vartheta x_{n,\vartheta}^{m_k(n)-1} \leq \vartheta x_{n,\vartheta}^{m_k(n)-1} \leq \mu_{k,n}m_k(n) \leq \frac{5}{12}\log\left(\frac{n}{\alpha(n)}\right)\alpha(n)$$

from Equation (2.4.4). Condition (3) now follows from

$$\left(\frac{n}{\alpha(n)}\right)^{\frac{5}{12}-2\delta}\log\left(\frac{n}{\alpha(n)}\right)\alpha(n) = o\left(n^{\frac{5}{12}}(\alpha(n))^{\frac{7}{12}}\right)$$

as $n \to \infty$, so we may apply Proposition 2.1.4. Thus, by Lemma 2.3.1,

$$\begin{aligned}
\text{(2.4.6)} \qquad \mathbb{E}\left[\prod_{k=1}^{K} \mathrm{e}^{s_k \widetilde{C}_{m_k}}\right] &\sim \frac{\exp\left(\sum_{k=1}^{K} \mu_{k,n}\right)}{Z^{(n)}} f_n\left(x_{n,\vartheta}\right) \\
&= \frac{\exp\left(\sum_{k=1}^{K} \mu_{k,n}\right)}{Z^{(n)}} \exp\left(\sum_{k=1}^{K}\left(\mathrm{e}^{s_k - \log(\mu_{k,n})} - 1\right)\mu_{k,n}\right) \\
&= \frac{\exp\left(\sum_{k=1}^{K} \mathrm{e}^{s_k}\right)}{Z^{(n)}}
\end{aligned}$$

as $n \to \infty$. Note that the normalizing constant depends on n. By setting $s_k = 0$ for all $1 \leq k \leq K$, we obtain

$$Z^{(n)} \xrightarrow{n\to\infty} \mathrm{e}^{K}. \tag{2.4.7}$$

Hence,

$$\lim_{n\to\infty} \mathbb{E}\left[\prod_{k=1}^{K} \mathrm{e}^{s_k \widetilde{C}_{m_k}}\right] = \prod_{k=1}^{K} \exp\left(\mathrm{e}^{s_k} - 1\right).$$

The claim then follows from Corollary 1.2.7. □

In particular, Proposition 2.4.4 allows us to draw conclusions concerning the probability that an untilted cycle count assumes a certain value.

COROLLARY 2.4.6. *Let $(m(n))_{n\in\mathbb{N}}$ be such that Equation (2.4.4) holds. Then we have*

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[C_{m(n)} = l\right] \sim \frac{\left(\mu_{m(n)}(n)\right)^{l}}{\mathrm{e}^{\mu_{m(n)}(n)} l!} \tag{2.4.8}$$

for each $l \in \mathbb{N}_0$. In particular, if $\lim_{n\to\infty} \mu_{m(n)}(n) = 0$, we have

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[C_{m(n)} = l\right] \sim \frac{\left(\mu_{m(n)}(n)\right)^{l}}{l!}$$

as $n \to \infty$.

PROOF. If a sequence $\left(C_{m(n)}^{(\mu_{m(n)}(n))}\right)_{n\in\mathbb{N}}$ of $\mathbb{N}_0$-valued random variables converges in distribution to another $\mathbb{N}_0$-valued random variable X, it follows that

$$\lim_{n\to\infty} \mathbb{P}\left[C_{m(n)}^{(\mu_{m(n)}(n))} = l\right] = \mathbb{P}\left[X = l\right]$$

for all $l \in \mathbb{N}_0$. By Proposition 2.4.4, on the one hand, we thus have

$$\lim_{n\to\infty} \mathbb{P}\left[C_{m(n)}^{(\mu_{m(n)}(n))} = l\right] = \frac{\mathrm{e}^{-1}}{l!}.$$

On the other hand, by Equations (2.4.2) and (2.4.7),

$$\mathbb{P}\left[C_{m(n)}^{(\mu_{m(n)}(n))} = l\right] \sim \mathrm{e}^{-1} \frac{\mathrm{e}^{\mu_{m(n)}(n)}}{\left(\mu_{m(n)}(n)\right)^{l}} \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[C_{m(n)} = l\right],$$

and the claim follows. □

REMARK 2.4.7. If $\mu_{m(n)}(n)$ diverges faster than in Equation (2.4.4), the problem arises whether the perturbation still fulfils condition (3) in Definition 2.1.2, which in turn depends on the power K with which condition (2) is satisfied. From a technical point of view, an alternative approach would be to include the dependence of the moment-generating function on the s_k into the array $\boldsymbol{q}$. Then, however, a problem arises when one analyzes the asymptotics provided by the saddle-point method. In Equation (2.4.6) in the proof of Proposition 2.4.4, the relevant terms λ_0 and $n\log\left(x_{n,\vartheta}\right)$ in the exponent are produced once with positive and once with negative sign so that the contributions exactly cancel each other out. In the new situation, one pair of the terms would be functions of the variables s_k. By setting $s_k = 0$, one could again determine the limit of the normalizing constants $Z^{(n)}$. But then one would have to study the behaviour of the respective

functions for different values of s_k without expanding them (since we want to show convergence to Poisson-distributed random variables). The instruments necessary for this step are currently lacking.
Nevertheless, the statement might still hold since it is consistent with the fact that asymptotic expected value and variance are also in this case of the same order (see Corollary 2.4.10). It is difficult to provide numerical evidence either way, however, because the strong parts of the statement deal with rare events in this case.

2.4.3. Diverging Expected Values. The present section deals with the limit distributions of $C_{m(n)}$ in the case of diverging $\mu_{m(n)}(n)$ (regime 4) and establishes a central limit theorem after suitable centering and rescaling. For technical reasons, we are going to rely on a certain additional assumption which we do not believe to be necessary. Proposition 2.4.8 gives the most general result provided in this thesis in this regard. Since the additional assumption does not lend itself to an immediate interpretation, Corollary 2.4.10 presents an instructive special case. The question of how to prove the result under weaker assumptions will be discussed in Remark 2.4.11.

PROPOSITION 2.4.8 ([**10**, Proposition 4.4]). *Let $m_k : \mathbb{N} \to \mathbb{N}$ for $1 \leq k \leq K$ such that $m_k(n) \leq \alpha(n)$ and $m_{k_1}(n) \neq m_{k_2}(n)$ if $k_1 \neq k_2$ for large n. If*

$$\mu_{m_k(n)}(n) \to \infty$$

and

$$n^{-\frac{5}{12}} \alpha(n)^{-\frac{7}{12}} \frac{x_{n,\vartheta}^{m_k(n)}}{\sqrt{\mu_{m_k(n)}(n)}} \to 0 \tag{2.4.9}$$

hold for all k, then we have

$$\left(\frac{C_{m_1(n)} - \mu_{m_1(n)}(n)}{\sqrt{\mu_{m_1(n)}(n)}}, \ldots, \frac{C_{m_K(n)} - \mu_{m_K(n)}(n)}{\sqrt{\mu_{m_K(n)}(n)}} \right) \xrightarrow{d} (N_1, \ldots, N_K)$$

as $n \to \infty$. Here, for each n, $\left(C_{m_1(n)}, \ldots, C_{m_K(n)}\right)$ are random variables under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$, and the N_k are independent standard normal distributed random variables.

REMARK 2.4.9. Even though Proposition 2.4.4 does in general not apply to the cycle lengths which are addressed by Proposition 2.4.8, the fact that expected value and variance of the cycle counts are asymptotically of the same order still hints at an underlying Poissonian structure. One might thus expect the cycle numbers in Proposition 2.4.8 to converge in the mod-Poisson sense, which would be a much stronger convergence result than that of Proposition 2.4.8, but we cannot prove it at present. The concepts of mod-Poisson and the related mod-Gaussian convergence have been introduced in [**40, 35**].

PROOF OF PROPOSITION 2.4.8. The start of the proof is similar to the proof of Proposition 2.4.1, but we have to be more careful that the conditions for applying Proposition 2.1.4 are met. Equation (1.3.6) yields

$$\begin{aligned} &\mathbb{E}_{n,\alpha}^{(\vartheta)} \left[\prod_{k=1}^{K} \exp\left(s_k \frac{C_{m_k(n)} - \mu_{m_k(n)}(n)}{\sqrt{\mu_{m_k(n)}(n)}} \right) \right] \\ =& \frac{1}{Z_{n,\alpha,\vartheta}} [z^n] \exp\left[\sum_{k=1}^{K} \left(\exp\left(\frac{s_k}{\sqrt{\mu_{m_k(n)}(n)}} \right) - 1 \right) \frac{\vartheta z^{m_k(n)}}{m_k(n)} + \sum_{j=1}^{\alpha(n)} \frac{\vartheta z^j}{j} - \sum_{k=1}^{K} s_k \sqrt{\mu_{m_k(n)}(n)} \right]. \end{aligned}$$

With the array $\boldsymbol{q}_\vartheta$ being admissible by Lemma 2.2.4, we are led to define

$$f_n(z) := \exp\left[\sum_{k=1}^{K} \left(\exp\left(\frac{s_k}{\sqrt{\mu_{m_k(n)}(n)}} \right) - 1 \right) \frac{\vartheta z^{m_k(n)}}{m_k(n)} \right] \exp\left(- \sum_{k=1}^{K} s_k \sqrt{\mu_{m_k(n)}(n)} \right).$$

Fix again $s_k \geq 0$ for $1 \leq k \leq K$. Then conditions (1) and (2) in Definition 2.1.2 hold because f_n is entire and

$$|f_n(z)| \leq |f_n(x_{n,\vartheta})|$$

for $|z| = x_{n,\vartheta}$. Concerning condition (3), note that

$$f_n'(z) = \sum_{k=1}^{K} \left(\exp\left(\frac{s_k}{\sqrt{\mu_{m_k(n)}(n)}} \right) - 1 \right) \vartheta z^{m_k(n)-1} f_n(z).$$

So

$$\begin{aligned} \|\!|f_n|\!\|_n \leq & n^{-\frac{5}{12}} (\alpha(n))^{-\frac{7}{12}} \sum_{k=1}^{K} \left(\exp\left(\frac{s_k}{\sqrt{\mu_{m_k(n)}(n)}} \right) - 1 \right) \vartheta x_{n,\vartheta}^{m_k(n)-1} \\ = & \mathcal{O}\left(n^{-\frac{5}{12}} (\alpha(n))^{-\frac{7}{12}} \sum_{k=1}^{K} \frac{x_{n,\vartheta}^{m_k(n)-1}}{\sqrt{\mu_{m_k(n)}(n)}} \right), \end{aligned}$$

and our assumption in Equation (2.4.9) ensures that the condition is satisfied. Hence, by Proposition 2.1.4 and Lemma 2.3.1,

$$\begin{aligned} & \mathbb{E}_{n,\alpha}^{(\vartheta)} \left[\prod_{k=1}^{K} \exp\left(s_k \frac{C_{m_k(n)} - \mu_{m_k(n)}(n)}{\sqrt{\mu_{m_k(n)}(n)}} \right) \right] \\ \sim & f_n(x_{n,\vartheta}) \\ = & \exp\left[\sum_{k=1}^{K} \left(\exp\left(\frac{s_k}{\sqrt{\mu_{m_k(n)}(n)}} \right) - 1 \right) \mu_{m_k(n)}(n) - \sum_{k=1}^{K} s_k \sqrt{\mu_{m_k(n)}(n)} \right] \\ = & \exp\left[\sum_{k=1}^{K} \left(s_k \sqrt{\mu_{m_k(n)}(n)} + \frac{s_k^2}{2} + \mathcal{O}\left(\frac{s_k^3}{\sqrt{\mu_{m_k(n)}(n)}} \right) \right) - \sum_{k=1}^{K} s_k \sqrt{\mu_{m_k(n)}(n)} \right] \\ = & \exp\left[\sum_{k=1}^{K} \frac{s_k^2}{2} + \mathcal{O}\left(\sum_{k=1}^{K} \frac{1}{\sqrt{\mu_{m_k(n)}(n)}} \right) \right]. \end{aligned}$$

Since, by assumption, the $\mu_{m_k(n)}(n)$ diverge as n tends to infinity, we conclude

$$\mathbb{E}_{n,\alpha}^{(\vartheta)} \left[\prod_{k=1}^{K} \exp\left(s_k \frac{C_{m_k(n)} - \mu_{m_k(n)}(n)}{\sqrt{\mu_{m_k(n)}(n)}} \right) \right] \xrightarrow{n\to\infty} \exp\left[\sum_{k=1}^{K} \frac{s_k^2}{2} \right].$$

The claim then follows from Corollary 1.2.7. □

COROLLARY 2.4.10 ([**10**, Theorem 2.4]). *Let $m_k : \mathbb{N} \to \mathbb{N}$ for $1 \leq k \leq K$ such that $m_k(n) \leq \alpha(n)$ and $m_{k_1}(n) \neq m_{k_2}(n)$ if $k_1 \neq k_2$ for large n. Further assume that*

$$\mu_{m_k(n)}(n) \to \infty$$

as $n \to \infty$ and that there is $\delta > 0$ with

$$\alpha(n) = \Omega\left(n^{\frac{1}{7}+\delta} \right) \tag{2.4.10}$$

for large n. Then

$$\left(\frac{C_{m_1(n)} - \mu_{m_1(n)}(n)}{\sqrt{\mu_{m_1(n)}(n)}}, \ldots, \frac{C_{m_K(n)} - \mu_{m_K(n)}(n)}{\sqrt{\mu_{m_K(n)}(n)}} \right) \xrightarrow{d} (N_1, \ldots, N_K)$$

holds as $n \to \infty$, where the N_k are independent standard normal distributed random variables.

PROOF. Since the corollary is supposed be a special case of Proposition 2.4.8, we only have to check that Equation (2.4.10) entails Equation (2.4.9) for all k. Then we may apply the proposition and conclude the claim. By Lemma 2.3.1, we have

$$\frac{x_{n,\vartheta}^{m_k(n)}}{\sqrt{\mu_{m_k(n)}(n)}} = \sqrt{\frac{m_k(n)\, x_{n,\vartheta}^{m_k(n)}}{\vartheta}} \leq \sqrt{\frac{\alpha(n)\, x_{n,\vartheta}^{\alpha(n)}}{\vartheta}} = \mathcal{O}\left(\sqrt{n \log(n)} \right).$$

So

$$n^{-\frac{5}{12}} (\alpha(n))^{-\frac{7}{12}} \frac{x_{n,\vartheta}^{m_k(n)}}{\sqrt{\mu_{m_k(n)}(n)}} = \mathcal{O}\left(n^{\frac{1}{12}} (\alpha(n))^{-\frac{7}{12}} \log(n) \right)$$

converges to 0 for all k due to Equation (2.4.10). Hence, Equation (2.4.9) holds for all k, and the claim follows. $\square$

REMARK 2.4.11. The additional assumptions in Equations (2.4.9) and (2.4.10) arise from the fact that our choice for the sequence $(f_n)_{n\in\mathbb{N}}$ has to be admissible in the sense of Definition 2.1.2. We do not believe them to be necessary because it is possible to consider the constant (with respect to z) functions

$$f_n(z) = \exp\left[-\sum_{k=1}^{K} s_k \sqrt{\mu_{m_k(n)}(n)}\right]$$

instead. The dependence of the moment-generating function on the arguments $(s_k)_{1\leq k\leq K}$ then necessitates that we consider a different triangular array $\boldsymbol{q}$ which is given by $q_{j,n} = \vartheta \exp\left(\frac{s_k}{\sqrt{\mu_{m_k(n)}(n)}}\right)$ for $j = m_k(n)$ and $q_{j,n} = \vartheta$ otherwise. One therefore needs to check that $\boldsymbol{q}$ is admissible in order to apply Proposition 2.1.4 in the new situation. A suitable Taylor expansion, which also requires information about the derivatives of the new saddle point(s) with respect to s_k, would then yield the desired result. Elaborating this outline is the subject of an ongoing master thesis project by Julian Mühlbauer under the supervision of Volker Betz and the author. Note also that such an approach is implemented in a different context in Section 2.7. We therefore expect Corollary 2.4.10 to hold for all sequences α satisfying Equation (2.0.2).

2.5. The Cycle Structure of Short Cycles

In this section we strengthen the results established in Propositions 2.4.1 and 2.4.4 for cycle numbers in the classical regime (cf. Section 2.3.2). It is a well-known fact (see Section 1.1.2.2) that the short cycles in ϑ-biased random permutations converge in total variation distance to independent Poisson-distributed random variables. Note that, since convergence is considered with respect to total variation distance, one is not restricted to consider a fixed number of cycle counts. Instead, one can look at all (or a selection of) cycle lengths up to any length $b(n)$ such that $b(n) = o(n)$ simultaneously. For the case of random permutations without macroscopic cycles, Theorem 2.5.1 states the same result as long as $b(n) = o(\alpha(n)/\log(n))$. Note that this threshold exactly coincides with the boundary of the classical regime of cycle lengths provided in Section 2.3.2. As a consequence of Theorem 2.5.1, the functional central limit theorem for cumulative cycle counts under the Ewens measure (see Equation (1.1.7)) also holds for the short cycles in constrained random permutations. This is the content of Corollary 2.5.7. We draw the conclusion that, in the limit of large n, imposing the constraint of a maximal cycle length $\alpha(n)$ does not affect the behaviour of the short cycles.
Theorem 2.5.1 has already been proved in [**10**] in the case of $\vartheta = 1$, whereas Corollary 2.5.7 has only been stated without proof as the fourth remark after [**10**, Theorem 2.6].
Throughout this section, let α be as in Equation (2.0.2).

2.5.1. Convergence in Total Variation Distance.

Recall that we denote the law of a random variable Z by $\mathcal{L}(Z)$.

THEOREM 2.5.1 ([**10**, Theorem 2.2]). *Let $b(n) = o\left(\frac{\alpha(n)}{\log(n)}\right)$. Then, as $n \to \infty$,*

$$d_{b(n)} := \left\| \mathcal{L}\left(\boldsymbol{Z}_{b(n)}\right) - \mathcal{L}\left(\boldsymbol{C}_{b(n)}\right) \right\|_{\mathrm{TV}} = \mathcal{O}\left(\frac{\alpha(n)}{n} + b(n)\frac{\log(n)}{\alpha(n)} \right).$$

Here, $\boldsymbol{C}_{b(n)} = (C_k)_{k=1}^{b(n)}$ are the cycle numbers under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ and $\boldsymbol{Z}_{b(n)} = (Z_k)_{k=1}^{b(n)}$ are independent Poisson-distributed random variables with parameters $\mathbb{E}[Z_k] = \frac{\vartheta}{k}$.

REMARK 2.5.2. Note that the upper bound for the rate of convergence in Theorem 2.5.1 may decrease arbitrarily slow for sequences b which increase comparatively fast since we only assume $b(n)\frac{\log(n)}{\alpha(n)} = o(1)$. Due to the term $\frac{\alpha(n)}{n}$ in the bound, we can show at most algebraic decay of $d_{b(n)}$ in n. These bounds are consistent with the discussion in Section 1.1.2.2 which states that, as far as is known, only uniform random permutations may exhibit exponential decay of the rate of convergence in this case.

In order to prove Theorem 2.5.1, we retrace and suitably modify the steps in the approach of [**4**], which deals with the classical case of uniform permutations. We will first derive an analogue of the conditioning relation for random permutations without macroscopic cycles and then give the main part of the proof of the theorem, which will be completed by several lemmata presented afterwards.
Let $\boldsymbol{C}_{b(n)}$ and $\boldsymbol{Z}_{b(n)}$ be as in the theorem. Recall that, by Equations (1.1.4) and (1.1.5), we have

$$T_{b_1 b_2} = \sum_{k=b_1+1}^{b_2} k Z_k$$

and

$$\mathbb{P}_n^{(\vartheta)}\left[\boldsymbol{C}_b = \boldsymbol{c}\right] = \mathbb{P}\left[\boldsymbol{Z}_b = \boldsymbol{c} \middle| T_{0n} = n\right]$$

for $\boldsymbol{c} \in \mathbb{N}_0^b$.
Lemma 2.5.3 derives an analogue of the conditioning relation for random permutations without macroscopic cycles.

LEMMA 2.5.3. *For $1 \leq b \leq \alpha(n)$ and $\boldsymbol{c} \in \mathbb{N}_0^b$,*

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\boldsymbol{C}_b = \boldsymbol{c}\right] = \mathbb{P}\left[\boldsymbol{Z}_b = \boldsymbol{c} \middle| T_{0\alpha(n)} = n\right] \tag{2.5.1}$$

holds.

Proof. By definition, we have

$$\mathbb{P}_{n,\alpha}^{(\vartheta)} = \mathbb{P}_{n}^{(\vartheta)} \left[\cdot \middle| S_{n,\alpha} \right].$$

So,

$$\begin{aligned}
\mathbb{P}_{n,\alpha}^{(\vartheta)} \left[\boldsymbol{C}_b = \boldsymbol{c} \right] =& \frac{\mathbb{P}_{n}^{(\vartheta)} \left[\{ \boldsymbol{C}_b = \boldsymbol{c} \} \cap \{ C_k = 0 : \alpha(n) < k \le n \} \right]}{\mathbb{P}_{n}^{(\vartheta)} \left[C_k = 0 : \alpha(n) < k \le n \right]} \\
=& \frac{\mathbb{P} \left[\{ \boldsymbol{Z}_b = \boldsymbol{c} \} \cap \{ Z_k = 0 : \alpha(n) < k \le n \} \middle| T_{0n} = n \right]}{\mathbb{P} \left[Z_k = 0 : \alpha(n) < k \le n \middle| T_{0n} = n \right]} \\
=& \frac{\mathbb{P} \left[\{ \boldsymbol{Z}_b = \boldsymbol{c} \} \cap \{ Z_k = 0 : \alpha(n) < k \le n \} \cap \{ T_{0n} = n \} \right]}{\mathbb{P} \left[\{ Z_k = 0 : \alpha(n) < k \le n \} \cap \{ T_{0n} = n \} \right]} \\
=& \frac{\mathbb{P} \left[\{ \boldsymbol{Z}_b = \boldsymbol{c} \} \cap \left\{ T_{0\alpha(n)} = n \right\} \cap \{ Z_k = 0 : \alpha(n) < k \le n \} \right]}{\mathbb{P} \left[\left\{ T_{0\alpha(n)} = n \right\} \cap \{ Z_k = 0 : \alpha(n) < k \le n \} \right]}.
\end{aligned}$$

Independence of $(Z_k)_k$ then yields

$$\begin{aligned}
\mathbb{P}_{n,\alpha}^{(\vartheta)} \left[\boldsymbol{C}_b = \boldsymbol{c} \right] =& \frac{\mathbb{P} \left[\{ \boldsymbol{Z}_b = \boldsymbol{c} \} \cap \left\{ T_{0\alpha(n)} = n \right\} \right]}{\mathbb{P} \left[T_{0\alpha(n)} = n \right]} \\
=& \mathbb{P} \left[\boldsymbol{Z}_b = \boldsymbol{c} \middle| T_{0\alpha(n)} = n \right],
\end{aligned}$$

which proves the claim. □

We can now give the main part of the

Proof of Theorem 2.5.1. Following [**4**], we compute for $\boldsymbol{c} \in \mathbb{N}_0^{b(n)}$ with $L(\boldsymbol{c}) := \sum_{k=1}^{b(n)} k c_k = r$ that

$$\begin{aligned}
\mathbb{P} \left[\boldsymbol{Z}_{b(n)} = \boldsymbol{c} \middle| T_{0\alpha(n)} = n \right] =& \frac{\mathbb{P} \left[\boldsymbol{Z}_{b(n)} = \boldsymbol{c}, T_{0\alpha(n)} = n \right]}{\mathbb{P} \left[T_{0\alpha(n)} = n \right]} \\
=& \frac{\mathbb{P} \left[\boldsymbol{Z}_{b(n)} = \boldsymbol{c}, T_{b(n)\alpha(n)} = n - r \right]}{\mathbb{P} \left[T_{0\alpha(n)} = n \right]} \\
=& \frac{\mathbb{P} \left[\boldsymbol{Z}_{b(n)} = \boldsymbol{c} \right] \mathbb{P} \left[T_{b(n)\alpha(n)} = n - r \right]}{\mathbb{P} \left[T_{0\alpha(n)} = n \right]}
\end{aligned}$$

by applying the independence of the Z_k. Recall that the total variation distance between two probability measures $\mathbb{P}$ and $\mathbb{P}'$ on a discrete space Ω is given by

$$\| \mathbb{P} - \mathbb{P}' \|_{\mathrm{TV}} = \sum_{\omega \in \Omega} \left(\mathbb{P} [\{\omega\}] - \mathbb{P}' [\{\omega\}] \right)_+ .$$

So, according to Equation (2.5.1), we have

$$\begin{aligned}
d_{b(n)} =& \sum_{\boldsymbol{c} \in \mathbb{N}_0^{b(n)}} \left(\mathbb{P} \left[\boldsymbol{Z}_{b(n)} = \boldsymbol{c} \right] - \mathbb{P} \left[\boldsymbol{Z}_{b(n)} = \boldsymbol{c} \middle| T_{0\alpha(n)} = n \right] \right)_+ \\
=& \sum_{r=0}^{\infty} \left(\sum_{\boldsymbol{c} : L(\boldsymbol{c}) = r} \mathbb{P} \left[\boldsymbol{Z}_{b(n)} = \boldsymbol{c} \right] \right) \left(1 - \frac{\mathbb{P} \left[T_{b(n)\alpha(n)} = n - r \right]}{\mathbb{P} \left[T_{0\alpha(n)} = n \right]} \right)_+ \\
=& \sum_{r=0}^{\infty} \mathbb{P} \left[T_{0b(n)} = r \right] \left(1 - \frac{\mathbb{P} \left[T_{b(n)\alpha(n)} = n - r \right]}{\mathbb{P} \left[T_{0\alpha(n)} = n \right]} \right)_+ .
\end{aligned}$$

Consequently, we obtain for any $\rho(n) > 0$ that

$$d_{b(n)} \le \mathbb{P} \left[T_{0b(n)} \ge \rho(n) b(n) + 1 \right] + \sum_{r=0}^{\rho(n) b(n)} \mathbb{P} \left[T_{0b(n)} = r \right] \left(1 - \frac{\mathbb{P} \left[T_{b(n)\alpha(n)} = n - r \right]}{\mathbb{P} \left[T_{0\alpha(n)} = n \right]} \right)_+ . \tag{2.5.2}$$

The last part of the proof is separately bounding the summands in Equation (2.5.2) for a suitable choice of $\rho(n)$. Lemma 2.5.5 below shows that, for $\rho(n) = \log(n)$,

$$\mathbb{P} \left[T_{0b(n)} \ge \rho(n) b(n) + 1 \right] \le \left(\frac{\log(n)}{\mathrm{e}\vartheta} \right)^{-\log(n)}$$

for large n, so the term decays faster than any power of n. Concerning the second term in Equation (2.5.2), we calculate

$$\begin{aligned}
&\sum_{r=0}^{\rho(n)b(n)} \mathbb{P}\left[T_{0b(n)}=r\right]\left(1-\frac{\mathbb{P}\left[T_{b(n)\alpha(n)}=n-r\right]}{\mathbb{P}\left[T_{0\alpha(n)}=n\right]}\right)_+ \\
\leq &\max_{1\leq r\leq \rho(n)b(n)}\left(1-\frac{\mathbb{P}\left[T_{b(n)\alpha(n)}=n-r\right]}{\mathbb{P}\left[T_{0\alpha(n)}=n\right]}\right)_+ .
\end{aligned}$$

Applying Lemma 2.5.6 below with $\rho(n)=\log(n)$ then proves the claim. □

In the following, we will prove Lemmata 2.5.5 and 2.5.6.
It is a well-known fact that the moment-generating function of a random variable Z which is Poisson-distributed with parameter $\beta>0$ is given by $\mathbb{E}\left[\mathrm{e}^{sZ}\right]=\mathrm{e}^{\beta(\mathrm{e}^s-1)}$. Accordingly, we have

$$\mathbb{E}\left[\mathrm{e}^{sZ_k}\right]=\mathrm{e}^{\frac{\vartheta}{k}(\mathrm{e}^s-1)}$$

for all k. By independence, we conclude

$$\mathbb{E}\left[\mathrm{e}^{sT_{0b}}\right]=\prod_{k=1}^{b}\mathbb{E}\left[\mathrm{e}^{skZ_k}\right]=\mathrm{e}^{\sum_{k=1}^{b}\frac{\vartheta}{k}\left(\mathrm{e}^{ks}-1\right)}. \tag{2.5.3}$$

Lemmata 2.5.4 and 2.5.5 slightly generalize results from [**4**] by including the parameter $\vartheta>0$.

LEMMA 2.5.4 ([**4**, Lemma 7]). *For $s\geq 0$, we have*

$$\log\left(\mathbb{E}\left[\mathrm{e}^{sT_{0b}}\right]\right)\leq \vartheta\mathrm{e}^{bs}.$$

PROOF. Starting from Equation (2.5.3), the claim follows from

$$\begin{aligned}
\log\left(\mathbb{E}\left[\mathrm{e}^{sT_{0b}}\right]\right)=&\sum_{k=1}^{b}\frac{\vartheta}{k}\left(\mathrm{e}^{ks}-1\right)\\
=&\vartheta\sum_{k=1}^{b}\int_0^s \mathrm{e}^{kv}\mathrm{d}v\\
\leq&\vartheta b\int_0^s \mathrm{e}^{bv}\mathrm{d}v\\
=&\vartheta\left(\mathrm{e}^{bs}-1\right).
\end{aligned}$$

□

LEMMA 2.5.5 ([**4**, Lemma 8]). *Let $\rho\geq\vartheta$. Then,*

$$\mathbb{P}\left[T_{0b}\geq \rho b\right]\leq\left(\frac{\rho}{\mathrm{e}\vartheta}\right)^{-\rho}.$$

PROOF. For any $s\geq 0$, we obtain

$$\mathbb{P}\left[T_{0b}\geq\rho b\right]\leq\mathbb{P}\left[\mathrm{e}^{sT_{0b}}\geq \mathrm{e}^{s\rho b}\right]\leq\frac{\mathbb{E}\left[\mathrm{e}^{sT_{0b}}\right]}{\mathrm{e}^{s\rho b}}$$

by monotonicity of the exponential function and Markov's inequality. Hence,

$$0<\mathbb{P}\left[T_{0b}\geq\rho b\right]\leq\inf_{s\geq 0}\frac{\mathbb{E}\left[\mathrm{e}^{sT_{0b}}\right]}{\mathrm{e}^{s\rho b}}$$

holds. By monotonicity and continuity of the logarithm, we conclude

$$\log\left(\inf_{s\geq 0}\frac{\mathbb{E}\left[\mathrm{e}^{sT_{0b}}\right]}{\mathrm{e}^{s\rho b}}\right)=\inf_{s\geq 0}\log\left(\frac{\mathbb{E}\left[\mathrm{e}^{sT_{0b}}\right]}{\mathrm{e}^{s\rho b}}\right).$$

Thus, by Lemma 2.5.4,

$$\begin{aligned}\log\left(\mathbb{P}\left[T_{0b}\geq\rho b\right]\right)\leq&\inf_{s\geq 0}\left(\vartheta\mathrm{e}^{bs}-s\rho b\right)\\ \leq&\inf_{t\geq 0}\left(\vartheta\mathrm{e}^{t}-t\rho\right)\\ =&\vartheta\inf_{t\geq 0}\left(\mathrm{e}^{t}-t\frac{\rho}{\vartheta}\right)\\ =&-\rho\log\left(\frac{\rho}{\mathrm{e}\vartheta}\right).\end{aligned}$$

The last step follows since, by strict convexity, $\mathrm{e}^t-t\frac{\rho}{\vartheta}$ attains its minimum when

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{t}-t\frac{\rho}{\vartheta}\right)\bigg|_{t=t_{min}}=\mathrm{e}^{t_{min}}-\frac{\rho}{\vartheta},$$

which is equivalent to $t_{min}=\log\left(\frac{\rho}{\vartheta}\right)$. By assumption, $t_{min}=\log\left(\frac{\rho}{\vartheta}\right)\geq 0$, so

$$\vartheta\inf_{t\geq 0}\left(\mathrm{e}^{t}-t\frac{\rho}{\vartheta}\right)=\vartheta\left(\mathrm{e}^{t_{min}}-t_{min}\frac{\rho}{\vartheta}\right)=-\rho\log\left(\frac{\rho}{\mathrm{e}\vartheta}\right),$$

and the claim is proved. □

LEMMA 2.5.6 ([**10**, Lemma 4.3]). *Let* $b(n)=o\left(\frac{\alpha(n)}{\log(n)}\right)$ *and* $\rho(n)=\mathcal{O}(\log(n))$. *Then,*

$$\max_{1\leq r\leq\rho(n)b(n)}\left(1-\frac{\mathbb{P}\left[T_{b(n)\alpha(n)}=n-r\right]}{\mathbb{P}\left[T_{0\alpha(n)}=n\right]}\right)_{+}=\mathcal{O}\left(\frac{\alpha(n)}{n}+b(n)\frac{\log(n)}{\alpha(n)}\right)\tag{2.5.4}$$

as $n\to\infty$.

PROOF. By Equation (2.5.3) and independence of the Z_k, one sees that the probability generating function of $T_{b_1b_2}$ is given by

$$\mathbb{E}\left[z^{T_{b_1b_2}}\right]=\mathbb{E}\left[\mathrm{e}^{\log(z)T_{b_1b_2}}\right]=\exp\left(\sum_{j=b_1+1}^{b_2}\frac{\vartheta}{j}\left(z^j-1\right)\right)\tag{2.5.5}$$

for $z>0$. By uniqueness of analytic continuation, Equation (2.5.5) holds for all $z\in\mathbb{C}$. Hence,

$$\mathbb{P}\left[T_{b(n)\alpha(n)}=m\right]=\left[z^m\right]\exp\left(\sum_{j=b(n)+1}^{\alpha(n)}\frac{\vartheta}{j}\left(z^j-1\right)\right)$$

for all $m\in\mathbb{N}_0$. We conclude

$$\begin{aligned}\mathbb{P}\left[T_{b(n)\alpha(n)}=n-r\right]=&\mathrm{e}^{-\sum_{j=b(n)+1}^{\alpha(n)}\frac{\vartheta}{j}}\left[z^{n-r}\right]\exp\left(\sum_{j=b(n)+1}^{\alpha(n)}\frac{\vartheta}{j}z^j\right)\\ =&\mathrm{e}^{-\sum_{j=b(n)+1}^{\alpha(n)}\frac{\vartheta}{j}}\left[z^{n}\right]z^r\exp\left(\sum_{j=b(n)+1}^{\alpha(n)}\frac{\vartheta}{j}z^j\right)\end{aligned}$$

and

$$\mathbb{P}\left[T_{0\alpha(n)}=n\right]=\mathrm{e}^{-\sum_{j=b(n)+1}^{\alpha(n)}\frac{\vartheta}{j}}\left[z^n\right]\exp\left(\sum_{j=1}^{b(n)}\frac{\vartheta}{j}\left(z^j-1\right)\right)\exp\left(\sum_{j=b(n)+1}^{\alpha(n)}\frac{\vartheta}{j}z^j\right).$$

We want to compute the quotient $\mathbb{P}\left[T_{b(n)\alpha(n)}=n-r\right]/\mathbb{P}\left[T_{0\alpha(n)}=n\right]$ in Equation (2.5.4). Thus, the prefactors $\mathrm{e}^{-\sum_{j=b(n)+1}^{\alpha(n)}\frac{\vartheta}{j}}$ will cancel each other out, and we have two terms to which we may apply Proposition 2.1.4. In both cases, the relevant triangular array $\boldsymbol{q}$ is given by $q_{j,n}=\vartheta$ if $j\geq b(n)+1$ and $q_{j,n}=0$ otherwise. We need to check admissibility of $\boldsymbol{q}$ in the sense of Definition 2.1.1. By definition, the saddle point $x_{n,\boldsymbol{q}}$ is the unique positive solution of

$$n=\sum_{j=b(n)+1}^{\alpha(n)}\vartheta x_{n,\boldsymbol{q}}^j.$$

Further, recall that $x_{n,\vartheta}$ is defined by

$$n = \sum_{j=1}^{\alpha(n)} \vartheta x_{n,\vartheta}^j$$

and let $\tilde{x}_{n,\vartheta}$ be the unique positive solution of

$$n = \sum_{j=1}^{\lfloor \alpha(n)/2 \rfloor} \vartheta \tilde{x}_{n,\vartheta}^j.$$

By $b(n) = o\left(\frac{\alpha(n)}{\log(n)}\right)$,

$$x_{n,\vartheta} \le x_{n,\boldsymbol{q}} \le \tilde{x}_{n,\vartheta} \tag{2.5.6}$$

holds for large n. Note that the definitions of $x_{n,\vartheta}$ and $\tilde{x}_{n,\vartheta}$ only differ in the upper bound of summation. Applying Lemma 2.3.1 to Equation (2.5.6) therefore yields

$$\log\left(\frac{n}{\alpha(n)}\right) \sim \alpha(n)\log(x_{n,\vartheta}) \le \alpha(n)\log(x_{n,\boldsymbol{q}}) \le 2\frac{\alpha(n)}{2}\log(\tilde{x}_{n,\vartheta}) \sim 2\log\left(\frac{n}{\alpha(n)}\right), \tag{2.5.7}$$

which implies condition (1). By Equations (2.1.3) and (2.5.6), we have

$$\begin{aligned}
n\alpha(n) \ge& \lambda_{2,n} \\
=& \sum_{j=b(n)+1}^{\alpha(n)} \vartheta j x_{n,\boldsymbol{q}}^j \\
\ge& \sum_{j=b(n)+1}^{\alpha(n)} \vartheta j x_{n,\vartheta}^j \\
=& \lambda_{2,n,\alpha,\boldsymbol{q}_\vartheta} - \sum_{j=1}^{b(n)} \vartheta j x_{n,\vartheta}^j \\
\ge& \lambda_{2,n,\alpha,\boldsymbol{q}_\vartheta} + nb(n).
\end{aligned}$$

Since $\lambda_{2,n,\alpha,\boldsymbol{q}_\vartheta} \sim n\alpha(n)$ by Lemma 2.3.1 and $nb(n) = o(n\alpha(n))$ by assumption, condition (2) follows. Note that condition (3) holds by construction, so $\boldsymbol{q}$ is admissible.
The relevant perturbations are given by

$$f_{1,n}(z) = z^r,$$

$1 \le r \le \rho(n)b(n)$, and

$$f_{2,n}(z) = \mathrm{e}^{\sum_{j=1}^{b(n)} \frac{\vartheta}{j}(z^j - 1)},$$

respectively. One easily sees that both $f_{1,n}$ and $f_{2,n}$ are entire. It is also clear that

$$|f_{i,n}(z)| \le |f_{i,n}(|z|)|$$

for $i = 1, 2$, so both $f_{1,n}$ and $f_{2,n}$ fulfil conditions (1) and (2) in Definition 2.1.2. An easy calculation then yields

$$\|\!|f_{1,n}|\!\|_n \le rn^{-\frac{5}{12}}(\alpha(n))^{-\frac{7}{12}} = \mathcal{O}\left(\frac{\log(n)\,b(n)}{n^{5/12}(\alpha(n))^{7/12}}\right)$$

uniformly in $1 \le r \le \rho(n)b(n)$ and

$$\|\!|f_{2,n}|\!\|_n \le n^{-\frac{5}{12}}(\alpha(n))^{-\frac{7}{12}} \sum_{j=0}^{b(n)-1} \vartheta x_{n,\boldsymbol{q}}^j \le \vartheta n^{-\frac{5}{12}}(\alpha(n))^{-\frac{7}{12}} b(n)\, x_{n,\boldsymbol{q}}^{b(n)}.$$

From Equation (2.5.7), it follows that

$$0 \le b(n)\log(x_{n,\boldsymbol{q}}) \le \frac{2b(n)}{\alpha(n)}\frac{\alpha(n)}{2}\log(\tilde{x}_{n,\vartheta}) \sim 2\frac{b(n)\log(n/\alpha(n))}{\alpha(n)} = o(1). \tag{2.5.8}$$

So

$$\lim_{n\to\infty} x_{n,\boldsymbol{q}}^{b(n)} = 1 \tag{2.5.9}$$

and

$$\|f_{2,n}\|_n = \mathcal{O}\left(\frac{b(n)}{n^{5/12}(\alpha(n))^{7/12}}\right)$$

hold. Hence, both $f_{1,n}$ and $f_{2,n}$ satisfy condition (3) and are admissible. Proposition 2.1.4 and Remark 2.1.5 then yield

$$\frac{\mathbb{P}\left[T_{b(n)\alpha(n)} = n - r\right]}{\mathbb{P}\left[T_{0\alpha(n)} = n\right]} = \frac{f_{1,n}(x_{n,\boldsymbol{q}})}{f_{2,n}(x_{n,\boldsymbol{q}})}\left(1 + \mathcal{O}\left(\frac{\alpha(n)}{n} + \frac{\log(n)\, b(n)}{n^{5/12}(\alpha(n))^{7/12}}\right)\right),$$

uniformly in $1 \leq r \leq \rho(n)\, b(n)$. Moreover,

$$0 \leq \log(f_{2,n}(x_{n,\boldsymbol{q}})) = \sum_{j=1}^{b(n)} \frac{\vartheta}{j}\left(x_{n,\boldsymbol{q}}^j - 1\right) = \vartheta \int_1^{x_{n,\boldsymbol{q}}} \sum_{j=0}^{b(n)-1} v^j \,\mathrm{d}v \leq \vartheta b(n)(x_{n,\boldsymbol{q}} - 1)\, x_{n,\boldsymbol{q}}^{b(n)}.$$

By Equations (2.5.8) and (2.5.9),

$$\begin{aligned} 0 \leq \log(f_{2,n}(x_{n,\boldsymbol{q}})) &\leq \vartheta b(n)(x_{n,\boldsymbol{q}} - 1)\, x_{n,\boldsymbol{q}}^{b(n)} \\ &\sim \vartheta b(n) \log(x_{n,\boldsymbol{q}})\, x_{n,\boldsymbol{q}}^{b(n)} \\ &= \mathcal{O}\left(b(n)\frac{\log(n)}{\alpha(n)}\right) \end{aligned}$$

follows. Thus,

$$1 \leq f_{2,n}(x_{n,\boldsymbol{q}}) = 1 + \mathcal{O}\left(b(n)\frac{\log(n)}{\alpha(n)}\right)$$

and we conclude

$$\frac{1}{f_{2,n}(x_{n,\boldsymbol{q}})} = 1 + \mathcal{O}\left(b(n)\frac{\log(n)}{\alpha(n)}\right).$$

Note that

$$f_{1,n}(x_{n,\boldsymbol{q}}) \geq 1,$$

which implies

$$\frac{f_{1,n}(x_{n,\boldsymbol{q}})}{f_{2,n}(x_{n,\boldsymbol{q}})} \geq 1 + \mathcal{O}\left(b(n)\frac{\log(n)}{\alpha(n)}\right),$$

where the error term is uniform in $1 \leq r \leq \rho(n)\, b(n)$. Hence,

$$\max_{1 \leq r \leq \rho(n) b(n)}\left(1 - \frac{\mathbb{P}\left[T_{b(n)\alpha(n)} = n - r\right]}{\mathbb{P}\left[T_{0\alpha(n)} = n\right]}\right)_+ = \mathcal{O}\left(\frac{\alpha(n)}{n} + b(n)\frac{\log(n)}{\alpha(n)}\right)$$

since $n^{-\frac{5}{12}}(\alpha(n))^{-\frac{7}{12}} = \mathcal{O}\left((\alpha(n))^{-1}\right)$, and the claim is proved. □

2.5.2. Functional Central Limit Theorem for Short Cycles. Having established convergence of the counts of short cycles of constrained permutations in total variation distance to independent Poisson-distributed random variables in Section 2.5.1 as in the classical case (cf. Section 1.1.2.2), we can now apply these results to show that the cumulative cycle counts of short cycles satisfy the same functional central limit theorem as in ϑ-biased random permutations (see 1.1.2.3). The proof rests on the triangle inequality and a special representation of the total variation distance.

COROLLARY 2.5.7. *Let a_1 be as in Equation (2.0.2) and $\delta > 0$. Then,*

$$\left(\frac{\sum_{j=1}^{\lfloor n^t \rfloor} C_j - \vartheta t \log(n)}{\sqrt{\vartheta \log(n)}}\right)_{t \in [0, a_1 - \delta]},$$

considered under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ for each n, converges in distribution to the standard Brownian motion in $\mathcal{D}[0, a_1 - \delta]$, where $\mathcal{D}[0, a_1 - \delta]$ is the space of càd-làg functions on $[0, a_1 - \delta]$ endowed with the Skorohod topology.

Corollary 2.5.7 is stated in the fourth remark after Theorem 2.6 in [**10**] for $\vartheta = 1$.

Proof of Corollary 2.5.7. The corollary is a consequence of Theorem 2.5.1. For each $n \in \mathbb{N}$, let $E_n := \mathbb{N}_0^{\lfloor n^{a_1-\delta} \rfloor}$ and endow it with the discrete topology. Further define the map $F_n : E_n \to \mathcal{D}[0, a_1 - \delta]$ by

$$F_n\left((c_j)_{1 \leq j \leq \lfloor n^{a_1-\delta} \rfloor} \right) = \left(\frac{\sum_{j=1}^{\lfloor n^t \rfloor} c_j - \vartheta t \log(n)}{\sqrt{\vartheta \log(n)}} \right)_{t \in [0, a_1 - \delta]}.$$

So F_n is continuous and thus also measurable (with respect to the Borel σ-algebras). Let $\mathbb{E}$ denote the expectation with respect to the Wiener measure on $\mathcal{D}[0, a_1 - \delta]$ and let $G : \mathcal{D}[0, a_1 - \delta] \to \mathbb{R}$ be continuous and bounded. Recall that $\boldsymbol{C}_{\lfloor n^{a_1-\delta} \rfloor} = (C_j)_{1 \leq j \leq \lfloor n^{a_1-\delta} \rfloor}$ by definition. By the transformation formula, we have to show that

$$\lim_{n\to\infty} \mathbb{E}_{n,\alpha}^{(\vartheta)} \left[G \circ F_n \left(\boldsymbol{C}_{\lfloor n^{a_1-\delta} \rfloor} \right) \right] = \mathbb{E}[G].$$

Since the cumulative cycle counts under the Ewens measure satisfy the functional central limit theorem (see Section 1.1.2.3), we already know that

$$\lim_{n\to\infty} \mathbb{E}_{n}^{(\vartheta)} \left[G \circ F_n \left(\boldsymbol{C}_{\lfloor n^{a_1-\delta} \rfloor} \right) \right] = \mathbb{E}[G].$$

In order to prove the corollary, it thus suffices to show that

$$\lim_{n\to\infty} \left| \mathbb{E}_{n}^{(\vartheta)} \left[G \circ F_n \left(\boldsymbol{C}_{\lfloor n^{a_1-\delta} \rfloor} \right) \right] - \mathbb{E}_{n,\alpha}^{(\vartheta)} \left[G \circ F_n \left(\boldsymbol{C}_{\lfloor n^{a_1-\delta} \rfloor} \right) \right] \right| = 0.$$

Let ν_n and ν_n' be the laws of $\boldsymbol{C}_{\lfloor n^{a_1-\delta} \rfloor}$ under $\mathbb{P}_n^{(\vartheta)}$ and $\mathbb{P}_{n,\alpha}^{(\vartheta)}$, respectively. Then the triangle inequality for the total variation distance, Theorem 2.5.1, and Section 1.1.2.2 entail that

$$\lim_{n\to\infty} \|\nu_n - \nu_n'\|_{TV} = 0.$$

Since E_n is countable, we have

$$\|\nu_n - \nu_n'\|_{TV} = \frac{1}{2} \sup_{f : E_n \to [-1,1]} \left| \int_{E_n} f \mathrm{d}\nu - \int_{E_n} f \mathrm{d}\nu' \right|$$

(see, e.g., [**42**, Proposition 4.5]). Note that $\frac{G \circ F_n}{\|G\|_\infty}$ maps E_n to $[-1, 1]$. Hence,

$$\begin{aligned}
&\left| \mathbb{E}_{n}^{(\vartheta)} \left[G \circ F_n \left(\boldsymbol{C}_{\lfloor n^{a_1-\delta} \rfloor} \right) \right] - \mathbb{E}_{n,\alpha}^{(\vartheta)} \left[G \circ F_n \left(\boldsymbol{C}_{\lfloor n^{a_1-\delta} \rfloor} \right) \right] \right| \\
&= \|G\|_\infty \left| \int_{E_n} \frac{G \circ F_n}{\|G\|_\infty} \mathrm{d}\nu - \int_{E_n} \frac{G \circ F_n}{\|G\|_\infty} \mathrm{d}\nu' \right| \\
&\leq 2 \|G\|_\infty \|\nu_n - \nu_n'\|_{TV} \xrightarrow{n\to\infty} 0,
\end{aligned}$$

and the claim is proved. $\square$

2.6. The Total Number of Cycles

In this section we prove a central limit theorem for the total number of cycles in random permutations without macroscopic cycles. We will consider more general sequences $(\alpha(n))_{n\in\mathbb{N}}$ than we do in the other sections, but then also give a specialized version of the result which holds for maximal cycle lengths satisfying Equation (2.0.2).

Recall that the total number of cycles C under $\mathbb{P}_n^{(\vartheta)}$ satisfies a central limit theorem (cf. Section 1.1.1). More precisely, by Equation (1.1.2), we have

$$\lim_{n\to\infty} \mathbb{P}_n^{(\vartheta)}\left[C - \vartheta\log(n) \le y\sqrt{\vartheta\log(n)}\right] = \Phi(y) \tag{2.6.1}$$

for all $y\in\mathbb{R}$, where Φ is the cumulative distribution function of the standard normal distribution. Note that, asymptotically, expected value and variance in the precise sense of Equation (2.6.1) are of the same order and grow logarithmically in n. Both features are best understood in light of the underlying Poissonian structure of the distribution of cycle counts (cf. Sections 1.1.2.1 and 1.1.2.2). While the total number of cycles still obeys a central limit theorem under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$, neither will expected value and variance be of the same order nor will any of them grow logarithmically if α satisfies Equation (2.0.2). Hence, the influence of the condition of a maximal cycle length $\alpha(n)$ on long cycles imposed in this case is strong enough to dominate the classical behaviour of short cycles (see Section 2.5), but the central limit theorem is still retained. A lower bound for the total number of cycles, which will turn out to be asymptotically equivalent to the expected value, is given by $\frac{n}{\alpha(n)}$. This feature is the first hint that most indices cluster in cycles of lengths close to $\alpha(n)$, an observation which will be quantified in Section 2.7.

We are aware of only two other instances in which the typical number of cycles does not grow logarithmically, but still satisfies a central limit theorem: These are given by cycle weights $\vartheta_j = j^\beta$ for some $\beta > 0$ in [**46**] and by the model of surrogate-spatial permutations [**15**], which deals with certain n-dependent cycle weights. In the latter case, under suitable assumptions, the total number of cycles grows linearly in n, so there is even a positive fraction of indices in cycles of finite lengths. Note that a central limit theorem for the total number of cycles has been proved in [**55**, Section 1.4] for $\vartheta = 1$ and constant sequences $\alpha(n) = \alpha(1)$. Indeed, [**55**] also covers random A-permutations with fixed finite set A of allowed cycle lengths satisfying certain additional assumptions.

The present section assembles the results and proofs already given in [**9**]. It transcends the paper by considering the conditioned Ewens measure for arbitrary $\vartheta > 0$ instead of only the constrained uniform measure. Moreover, the proofs directly employ moment-generating functions and thereby avoid the detour of probability-generating functions. The proofs are further simplified by considering a different auxiliary function $h_{n,C}$ than in [**9**].

2.6.1. Results. We will show that, in the precise sense of Theorem 2.6.1, asymptotic expected value and variance of the total number of cycles C under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ are given by

$$m_{n,\alpha} := \sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \text{ and } v_{n,\alpha} = m_{n,\alpha} - n^2\left(\sum_{j=1}^{\alpha(n)} j\vartheta x_{n,\vartheta}^j\right)^{-1} \tag{2.6.2}$$

for a wide range of possible sequences α.

Theorem 2.6.1 ([**9**, Theorem 2.1]). *Assume that*

$$\liminf_{n\to\infty} \alpha(n) \ge 4$$

and

$$\limsup_{n\to\infty} \frac{\alpha(n)}{n}\log(n)(\log\log(n))^2 < \frac{1}{12\vartheta\pi^2\mathrm{e}}. \tag{2.6.3}$$

Then, for all $y\in\mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[C - m_{n,\alpha} \le y\sqrt{v_{n,\alpha}}\right] = \Phi(y).$$

Here, Φ *is the cumulative distribution function of the standard normal distribution.*

Remark 2.6.2. The extra term in the definition of $v_{n,\alpha}$ in Equation (2.6.2) indicates that expected value and variance need not be of the same order.

If we only consider algebraically growing functions α, we can give asymptotic expansions of $m_{n,\alpha}$ and $v_{n,\alpha}$, respectively. We will therefore recapitulate the concept of asymptotic expansion (cf., e.g., [**49**]). Let $(\phi_j)_{j\in\mathbb{N}}$ be a sequence of functions which satisfies

$$\phi_{j+1}(z) = o(\phi_j(z)) \tag{2.6.4}$$

for all $j \in \mathbb{N}_0$ as $z \to \infty$. The expression $\sum_{j=0}^{\infty} a_j \phi_j(z)$ for $a_j \in \mathbb{C}$ is called an asymptotic expansion of the function $f(z)$ as $z \to \infty$, written as

$$f(z) \sim \sum_{j=0}^{\infty} a_j \phi_j(z),$$

if for all $n \in \mathbb{N}_0$ we have

$$f(z) = \sum_{j=0}^{n} a_j \phi_j(z) + o(\phi_n(z))$$

as $z \to \infty$. Note that, if an asymptotic expansion of a function with respect to a given sequence $(\phi_j)_{j\in\mathbb{N}}$ exists, it is unique due to Equation (2.6.4).
In order to state the theorem, we also need to introduce a special function well-known from analytic number theory (cf., e.g., [**61**, Chapter 5.4]). Let $\xi = \xi(u)$ be defined as the non-zero solution of the equation

$$\exp(\xi) = 1 + u\xi \tag{2.6.5}$$

if $u > 1$ with $\xi(1) = 0$. For $u > 1$ we have $\log(u) < \xi(u) \leq 2\log(u)$ (see [**45**]). Precise asymptotics will be provided in Lemma 2.6.9.

THEOREM 2.6.3 ([**9**, Theorem 2.2]). *Let α be as in Equation (2.0.2). Then the asymptotic expected value and variance defined in Equation (2.6.2) have the asymptotic expansions*

$$m_{n,\alpha} \sim \frac{n}{\alpha(n)} \sum_{j=0}^{\infty} \frac{j!}{\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right)^j} \tag{2.6.6}$$

and

$$v_{n,\alpha} \sim \frac{n}{\alpha(n)} \sum_{j=2}^{\infty} \frac{j!-1}{\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right)^j}, \tag{2.6.7}$$

respectively. In particular,

$$m_{n,\alpha} \sim \frac{n}{\alpha(n)} \text{ and } v_{n,\alpha} \sim \frac{n}{\alpha(n)} \frac{1}{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2}$$

hold.

While the central limit theorem for the total number of cycles is retained in the model of random permutations without macroscopic cycles, expected value and variance are now of different order. One should also note that both asymptotic mean and variance are not logarithmic in the system size which indicates the strength of the constraint. Indeed, we have $m_{n,\alpha} \sim \frac{n}{\alpha(n)}$ where $\frac{n}{\alpha(n)}$ is the theoretical minimum of needed cycles in our model (we have to distribute n indices among cycles with lengths less than or equal to $\alpha(n)$, so at least $\frac{n}{\alpha(n)}$ cycles are required).

2.6.2. Overview and Proofs.

The paper [**9**] relies on a criterion for probability-generating functions of sequences of integer-valued random variables to satisfy a central limit theorem (cf. [**55**, Theorem 4.2]), which is an application of Curtiss' Theorem (see Proposition 1.2.4). By Corollary 1.2.7, we may directly consider the corresponding moment-generating functions for only non-negative arguments, thereby simplifying some of the proofs and making the whole approach more transparent.

Let $(\gamma(n))_{n\in\mathbb{N}}$ be a sequence and recall that C denotes the total number of cycles. Then, by Equation (1.3.6), we have

$$(2.6.8)\qquad M_{n,C}\left(\frac{s}{\gamma(n)}\right) := \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\mathrm{e}^{\frac{s}{\gamma(n)}C}\right] = \frac{1}{Z_{n,\alpha,\vartheta}}\left[z^n\right]\exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta\mathrm{e}^{\frac{s}{\gamma(n)}}}{j}z^j\right).$$

Since the random variable C (under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$) is going to diverge with n, we assume for the rescaling that $\gamma(n)\to\infty$ as n goes to infinity. With the array $\boldsymbol{q}$ being given by $q_{j,n}=\vartheta\mathrm{e}^{\frac{s}{\gamma(n)}}$, denote the corresponding saddle point by

$$x_{n,C}(s) := x_{\boldsymbol{q}}.$$

Recall that it is defined as the unique positive solution of

$$(2.6.9)\qquad n=\sum_{j=1}^{\alpha(n)}\vartheta\mathrm{e}^{\frac{s}{\gamma(n)}}\left(x_{n,C}(s)\right)^j.$$

The asymptotics of the moment-generating function can again be determined by the saddle-point method.

PROPOSITION 2.6.4. *Under the assumptions of Theorem 2.6.1, we have for all $s\geq 0$ that*

$$\left[z^n\right]\exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta\mathrm{e}^{\frac{s}{\gamma(n)}}}{j}z^j\right) = \frac{\exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta\mathrm{e}^{\frac{s}{\gamma(n)}}}{j}\left(x_{n,C}(s)\right)^j\right)}{\left(x_{n,C}(s)\right)^n\sqrt{2\pi\sum_{j=1}^{\alpha(n)}\vartheta\mathrm{e}^{\frac{s}{\gamma(n)}}j\left(x_{n,C}(s)\right)^j}}\left(1+\mathcal{O}\left(\frac{\alpha(n)}{n}\right)\right)$$

as $n\to\infty$, where the error term is pointwise in s.

PROOF. Note that, for sequences α satisfying Equation (2.0.2), one can check the admissibility of $\boldsymbol{q}$ and then apply Proposition 2.6.4 to derive the statement. Since, however, the weights $q_{j,n}=\vartheta\mathrm{e}^{\frac{s}{\gamma(n)}}$ do not depend on the index j, the proof of Theorem 2 in [**45**] (which considers weights $q_{j,n}=1$) works almost verbatim for our situation which only takes large values of n into account. Hence, we can adopt the more general assumptions used there for α. In some respects the role of n is then played by $n'=n/\left(\vartheta\mathrm{e}^{\frac{s}{\gamma(n)}}\right)$ and Equation (2.6.3) ensures that

$$\alpha(n)\leq\frac{1}{12\pi^2\mathrm{e}}\frac{n}{\mathrm{e}^{\frac{s}{\gamma(n)}}\log(n)\left(\log\log(n)\right)^2}$$

for large n. See also [**9**, Proposition 3.2]. □

Let

$$(2.6.10)\qquad h_{n,C}(s):=\sum_{j=1}^{\alpha(n)}\frac{\vartheta\mathrm{e}^{\frac{s}{\gamma(n)}}}{j}\left(x_{n,C}(s)\right)^j-n\log\left(x_{n,C}(s)\right),$$

which is considerably simpler than the auxiliary function $h_{n,\alpha}$ in [**9**, Equation (3.4)]. Note that $x_{n,C}(s)=x_{n,\alpha}\left(\frac{n\mathrm{e}^{-s/\gamma(n)}}{\alpha(n)\vartheta}\right)$, so by Lemma 2.2.1 we obtain

$$\sum_{j=1}^{\alpha(n)}\vartheta\mathrm{e}^{\frac{s}{\gamma(n)}}j\left(x_{n,C}(s)\right)^j\sim n\alpha(n)$$

pointwise in s. By Equation (2.6.8) and Proposition 2.6.4, we arrive at

$$(2.6.11)\qquad M_{n,C}\left(\frac{s}{\gamma(n)}\right)=\frac{1}{Z_{n,\alpha,\vartheta}}\frac{\exp\left(h_{n,C}(s)\right)}{\sqrt{2\pi n\alpha(n)}}\left(1+o(1)\right),$$

where the error term is pointwise in s. In order to prove the theorems, we will expand the functions $h_{n,C}$ about $s=0$. Note that, due to $M_{n,C}(0)=1$ according to the definition of the moment-generating function, we have

$$(2.6.12)\qquad \frac{1}{Z_{n,\alpha,\vartheta}}\frac{\exp\left(h_{n,C}(0)\right)}{\sqrt{2\pi n\alpha(n)}}\xrightarrow{n\to\infty}1.$$

The natural next step is calculating the derivatives of $h_{n,C}$, which is done in Section 2.6.3 (see, in particular, Lemmata A.1.3 and A.1.4). We obtain that

$$h'_{n,C}(0) = \frac{1}{\gamma(n)} \sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^{j},$$

$$h''_{n,C}(0) = \frac{1}{(\gamma(n))^2} \sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^{j} + \frac{n}{\gamma(n)} \frac{x'_{n,C}(0)}{x_{n,\vartheta}} \tag{2.6.13}$$

$$= \frac{1}{(\gamma(n))^2} \left[\sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^{j} - n^2 \left(\sum_{j=1}^{\alpha(n)} j\vartheta x_{n,\vartheta}^{j} \right)^{-1} \right], \text{ and} \tag{2.6.14}$$

$$h'''_{n,C}(s) = \mathcal{O}\left(\frac{h''_n(0)}{\gamma(n)} \right) + \mathcal{O}\left(\frac{n}{\alpha(n)(\gamma(n))^3} \right), \tag{2.6.15}$$

where Equation (2.6.15) holds locally uniformly in $s \geq 0$. In order to prove the central limit theorem, we have to find a diverging sequence $(\gamma(n))_{n\in\mathbb{N}}$ such that both

$$h''_{n,C}(0) \to 1$$

and

$$h'''_{n,C}(s) = o(1), \tag{2.6.16}$$

locally uniformly in $s \geq 0$, are satisfied as $n \to \infty$ since in this case, we have

$$\begin{aligned} \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\mathrm{e}^{\frac{s}{\gamma(n)}(C - m_{n,\alpha})} \right] &\sim \exp\left(h'_{n,C}(0)\, s + \frac{1}{2} h''_{n,C}(0)\, s^2 + o(1) \right) \exp\left(-h'_{n,C}(0)\, s \right) \\ &\to \exp\left(\frac{s^2}{2} \right) \end{aligned}$$

by Equations (2.6.8), (2.6.11), and (2.6.12). Corollary 1.2.7 then entails the weak convergence stated in Theorem 2.6.1.
The natural choice is

$$\gamma(n) = \sqrt{ \sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^{j} - n^2 \left(\sum_{j=1}^{\alpha(n)} j\vartheta x_{n,\vartheta}^{j} \right)^{-1} }. \tag{2.6.17}$$

The main difficulty lies in the following: As will be seen below, the terms

$$\sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^{j} \geq \frac{1}{\alpha(n)} \sum_{j=1}^{\alpha(n)} \vartheta x_{n,\vartheta}^{j} = \frac{n}{\alpha(n)}$$

and

$$-n^2 \left(\sum_{j=1}^{\alpha(n)} j\vartheta x_{n,\vartheta}^{j} \right)^{-1} \sim -\frac{n}{\alpha(n)}$$

in Equation (2.6.14) are of the same leading order, but have different signs. So we need a careful analysis of the respective asymptotics to understand, on the one hand, the behaviour of the asymptotic variance and to ascertain that, on the other hand, $\gamma(n)$ grows fast enough so that Equation (2.6.16) holds. The relevant asymptotics will be given in Propositions 2.6.6 and 2.6.8 for the case of sequences α as in Theorem 2.6.1. As an intermediary step, however, we first state Lemma 2.6.5 which provides further information necessary for the interpretation of the results to follow.

Lemma 2.6.5 ([**9**, Lemma 4.2]). *Let $s \geq 0$ and $\alpha : \mathbb{N} \to \mathbb{N}$ be such that $n/(\vartheta\alpha(n)) > 3$ for large n. Then,*

$$\alpha(n) \log(x_{n,C}(s)) \to \infty$$

as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \frac{\alpha(n)}{n} = 0.$$

In this case,

$$\alpha(n)\log(x_{n,C}(s)) \approx \log\left(\frac{n}{\alpha(n)}\right) \tag{2.6.18}$$

holds locally uniformly in $s \geq 0$ and, if also $\alpha(n) \geq \log(n)$ for large n, we further have

$$\alpha(n)(x_{n,\vartheta} - 1) \approx \log\left(\frac{n}{\alpha(n)}\right). \tag{2.6.19}$$

PROOF. Note that

$$x_{n,C}(s) = x_{n,\alpha}\left(\frac{\mathrm{e}^{-s/\gamma(n)}}{\vartheta}\frac{n}{\alpha(n)}\right),$$

so the equivalence and Equation (2.6.18) follow directly from Equation (2.2.2) since $\frac{n}{\vartheta \mathrm{e}^{s/\gamma(n)}\alpha(n)} \geq 3$ for large n. Since $x - 1 \geq \log(x)$ for $x \geq 1$, it only remains to show the upper bound in order to prove Equation (2.6.19). By Lemma 2.2.1,

$$\begin{aligned}\alpha(n)(x_{n,\vartheta} - 1) =& \alpha(n)\left[\exp(\log(x_{n,\vartheta})) - 1\right] \\ \leq& \alpha(n)\left[\exp\left(2\frac{\log\left(\frac{n}{\vartheta\alpha(n)}\right)}{\alpha(n)}\right) - 1\right] \\ =& \mathcal{O}\left(\log\left(\frac{n}{\alpha(n)}\right)\right)\end{aligned}$$

for large n, and the claim follows. □

In order to prove Proposition 2.6.6 and Theorem 2.6.3, we will need the asymptotics of a special function: The exponential integral [**21**, Equation (6.2.5)] is defined by

$$\mathrm{Ei}(x) := \text{p.v.}\int_{-\infty}^{x}\frac{\exp(v)}{v}\mathrm{d}v, \tag{2.6.20}$$

where "p.v." stands for principal value, and has the asymptotic expansion [**21**, Equation (6.12.2)]

$$\mathrm{Ei}(x) \sim \frac{\exp(x)}{x}\sum_{j=0}^{\infty}\frac{j!}{x^j} \tag{2.6.21}$$

as $x \to \infty$.

PROPOSITION 2.6.6 ([**9**, Proposition 4.8]). *Let $\alpha : \mathbb{N} \to \mathbb{N}$ such that*

$$\lim_{n\to\infty}\frac{\alpha(n)}{n} = 0$$

and $\alpha(n) \geq 3$ for large n. Then,

$$\begin{aligned}\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}x_{n,\vartheta}^{j} =& \vartheta\log(\alpha(n)) + \frac{n}{\alpha(n)} + \frac{n}{(\alpha(n))^2(x_{n,\vartheta} - 1)} \\ &+ 2\frac{n}{(\alpha(n))^3(x_{n,\vartheta} - 1)\log(x_{n,\vartheta})} \\ &+ p_\alpha(n) + \mathcal{O}\left(\frac{n}{(\alpha(n))^4(x_{n,\vartheta} - 1)(\log(x_{n,\vartheta}))^2}\right),\end{aligned}$$

where $p_\alpha : \mathbb{N} \to \mathbb{R}$ is a non-negative function which fulfils $p_\alpha(n) = \mathcal{O}\left(\frac{x_{n,\vartheta}^{\alpha(n)}}{(\alpha(n))^3\log(x_{n,\vartheta})}\right)$.

REMARK 2.6.7. Proposition 2.6.6 indicates that some change of behaviour occurs when $\vartheta\log(\alpha(n))$ surpasses $\frac{n}{\alpha(n)}$ and becomes the dominating term. This blends in nicely with the fact that the classical uniform model has asymptotic expectation and variance of $\vartheta\log(n)$ and further corroborates the intuition that the constrained model more and more resembles the classical model when $\alpha(n)$ grows faster in n. It should be noted, however, that we do not prove the central limit theorem for such α (cf. the assumption in Equation (2.6.3)).

Proposition 2.6.8 ([**9**, Proposition 4.9]). *Let $\alpha : \mathbb{N} \to \mathbb{N}$ be such that*

$$\lim_{n\to\infty} \frac{\alpha(n)}{n} = 0.$$

Then,

$$-n^2 \left(\sum_{j=1}^{\alpha(n)} j\vartheta x_{n,\vartheta}^j \right)^{-1} = -\frac{n}{\alpha(n)} \sum_{j=0}^{\infty} \left(\frac{\alpha(n)}{n} \vartheta x_{n,\vartheta} + \alpha(n)(x_{n,\vartheta}-1) \right)^{-j} + \mathcal{O}(1) \tag{2.6.22}$$

as $n \to \infty$.
If also $\alpha(n) \geq 2$ for large n, we have

$$\begin{aligned} &- n^2 \left(\sum_{j=1}^{\alpha(n)} j\vartheta x_{n,\vartheta}^j \right)^{-1} \\ &= - \frac{n}{\alpha(n)} \sum_{j=0}^{2} \left(\frac{1}{\alpha(n)(x_{n,\vartheta}-1)} \right)^j + o\left(\frac{n}{(\alpha(n))^3 (x_{n,\vartheta}-1)^2} \right) + \mathcal{O}(1). \end{aligned} \tag{2.6.23}$$

The proofs of Propositions 2.6.6 and 2.6.8 are given in Section 2.6.3. We can now provide the

Proof of Theorem 2.6.1. The assumption in Equation (2.6.3) entails that

$$\frac{\alpha(n)}{n} \to 0$$

as n tends to infinity. Propositions 2.6.6 and 2.6.8 in combination with Equation (2.6.17) thus yield

$$h_n''(0) \xrightarrow{n\to\infty} 1$$

and

$$\begin{aligned} (\gamma(n))^2 =& \vartheta \log(\alpha(n)) + \frac{n}{\alpha(n)} + \frac{n}{(\alpha(n))^2 (x_{n,\vartheta}-1)} + 2\frac{n}{(\alpha(n))^3 (x_{n,\vartheta}-1) \log(x_{n,\vartheta})} \\ &+ p_\alpha(n) + \mathcal{O}\left(\frac{n}{(\alpha(n))^4 (x_{n,\vartheta}-1)(\log(x_{n,\vartheta}))^2} \right) \\ &- \frac{n}{\alpha(n)} \sum_{j=0}^{2} \left(\frac{1}{\alpha(n)(x_{n,\vartheta}-1)} \right)^j + o\left(\frac{n}{(\alpha(n))^3 (x_{n,\vartheta}-1)^2} \right) + \mathcal{O}(1), \end{aligned}$$

where $p_\alpha(n)$ is non-negative. Since

$$x - 1 \geq \log(x)$$

for $x \geq 1$ and $\alpha(n) \log(x_{n,\vartheta}) \to \infty$ by Lemma 2.6.5, we have

$$\begin{aligned} (\gamma(n))^2 \geq& \frac{n}{(\alpha(n))^3 (x_{n,\vartheta}-1) \log(x_{n,\vartheta})} \\ &+ o\left(\frac{n}{(\alpha(n))^3 (x_{n,\vartheta}-1) \log(x_{n,\vartheta})} \right) + \mathcal{O}(1). \end{aligned}$$

By Lemma 2.6.5,

$$\alpha(n) \log(x_{n,\vartheta}) \approx \log\left(\frac{n}{\alpha(n)} \right).$$

In order to proceed, we need to distinguish two cases: If $\alpha(n) \geq \log(n)$, we also have

$$\alpha(n)(x_{n,\vartheta}-1) \approx \log\left(\frac{n}{\alpha(n)} \right)$$

by Lemma 2.6.5, so Equation (2.6.16) is fulfilled in this case. If $\alpha(n) < \log(n)$,

$$\begin{aligned} \alpha(n)(x_{n,\vartheta}-1) \leq& \alpha(n) x_{n,\vartheta} \\ \leq& \alpha(n) \left(\frac{n}{\vartheta} \right)^{\frac{1}{\alpha(n)}} \end{aligned}$$

holds by Equation (2.2.5). Since $\alpha(n) \geq 4$ for large n, Equation (2.6.16) is also satisfied in the second case. The theorem is proved. $\square$

Having proved the general case, we now address Theorem 2.6.3. We will start by investigating the function ξ. Recall that Equation (2.6.5) defines $\xi(u)$ as the non-zero solution of

$$\exp(\xi(u)) = 1 + u\xi(u)$$

for $u > 1$. We assemble two auxiliary lemmata whose proofs may be found in [**45**].

LEMMA 2.6.9 ([**45**],[**10**, Lemma 4.11]). *Let $\alpha : \mathbb{N} \to \mathbb{N}$ such that $\log\left(\frac{n}{\vartheta\alpha(n)}\right) \leq \alpha(n) < \frac{n}{\vartheta}$ for all n. Then, pointwise in $\vartheta > 0$,*

$$x_{n,\vartheta} = \exp\left(\frac{\xi\left(\frac{n}{\vartheta\alpha(n)}\right)}{\alpha(n)}\right) + \mathcal{O}\left(\frac{\log\left(\frac{n}{\alpha(n)}+1\right)}{(\alpha(n))^2}\right),$$

$$\xi\left(\frac{n}{\vartheta\alpha(n)}\right) = \log\left(\frac{n}{\vartheta\alpha(n)}\right) + \log\left(\log\left(\frac{n}{\vartheta\alpha(n)}+2\right)\right) + \mathcal{O}\left(\frac{\log\left(\log\left(\frac{n}{\alpha(n)}+2\right)\right)}{\log\left(\frac{n}{\alpha(n)}+2\right)}\right),$$

and

$$\alpha(n)\log(x_{n,\vartheta}) = \xi\left(\frac{n}{\vartheta\alpha(n)}\right) + \mathcal{O}\left(\frac{\log\left(\frac{n}{\alpha(n)}+1\right)}{\alpha(n)}\right)$$

hold for large n.

The first and second parts of Lemma 2.6.9 reformulate statements in Lemmata 10 and 6 in [**45**] and entail the third part. For a smaller range of possible α, the third statement has already been given in an earlier version of [**45**].

LEMMA 2.6.10 ([**45**, Lemma 11]). *Let $K > 0$ and consider*

$$T_K(z) := \int_0^z \frac{\exp(v)-1}{v}\left(\frac{v}{K}\frac{\exp\left(\frac{v}{K}\right)}{\exp\left(\frac{v}{K}\right)-1} - 1\right)\mathrm{d}v. \tag{2.6.24}$$

If $0 \leq z \leq \pi K$, then

$$\left|T_K(z) + \frac{z}{2K}\right| \leq \frac{4\exp(z)}{K}.$$

We can now provide the

PROOF OF THEOREM 2.6.3. We only have to calculate the asymptotics of

$$m_{n,\alpha} = \sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j$$

and

$$v_{n,\alpha} = \sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j - n^2\left(\sum_{j=1}^{\alpha(n)} j\vartheta x_{n,\vartheta}^j\right)^{-1}.$$

In a similar way as in the proof of Proposition 2.6.6 below, we conclude that

$$\begin{aligned}\sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j &= \sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} + \vartheta\int_1^{x_{n,\vartheta}} \sum_{j=1}^{\alpha(n)} w^{j-1}\mathrm{d}w \\ &= \sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} + \vartheta\int_0^{\alpha(n)\log(x_{n,\vartheta})} \frac{\exp(v)-1}{v}\frac{v}{\alpha(n)}\frac{\exp\left(\frac{v}{\alpha(n)}\right)\mathrm{d}v}{\exp\left(\frac{v}{\alpha(n)}\right)-1},\end{aligned}$$

where the second line applies

$$\sum_{j=1}^{\alpha(n)} w^{j-1} = \frac{w^{\alpha(n)}-1}{w-1}$$

and the substitution $v = \alpha(n)\log(w)$. Since $\sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} = \mathcal{O}(\log(\alpha(n)))$ is of lower order by Equation (2.0.2), the term relevant to the asymptotic expansions in Equations (2.6.6) and (2.6.7) is

$$\begin{aligned}
&\vartheta \int_0^{\alpha(n)\log(x_{n,\vartheta})} \frac{\exp(v)-1}{v} \frac{v}{\alpha(n)} \frac{\exp\left(\frac{v}{\alpha(n)}\right)\mathrm{d}v}{\exp\left(\frac{v}{\alpha(n)}\right)-1}\\
=&\vartheta T_{\alpha(n)}\left(\alpha(n)\log(x_{n,\vartheta})\right) + \vartheta I\left(\alpha(n)\log(x_{n,\vartheta})\right).
\end{aligned}$$

Here, T is defined as in Lemma 2.6.10 and we further define

$$I(z) := \int_0^z \frac{\exp(v)-1}{v}\mathrm{d}v.$$

Note that $0 \leq \alpha(n)\log(x_{n,\vartheta}) \leq \pi\alpha(n)$ for large n by Lemma 2.6.5 and the growth condition for α in Equation (2.0.2). So we can apply Lemma 2.6.10, the important terms being

$$\frac{\alpha(n)\log(x_{n,\vartheta})}{2\alpha(n)} = \frac{\log(x_{n,\vartheta})}{2} \longrightarrow 0$$

by Lemma 2.6.5 and

$$\frac{4\exp\left(\alpha(n)\log(x_{n,\vartheta})\right)}{\alpha(n)} = 4\frac{x_{n,\vartheta}^{\alpha(n)}}{\alpha(n)} = \mathcal{O}\left(\frac{n}{(\alpha(n))^2}\log\left(\frac{n}{\alpha(n)}\right)\right)$$

by Lemma 2.3.1. Due to Equation (2.0.2), both terms do not contribute to the asymptotic expansions in Equations (2.6.6) and (2.6.7).
Note that the integrand

$$\frac{\exp(v)-1}{v}$$

in I is strictly increasing in $v \geq 0$. By Lemma 2.6.9, we have

$$\alpha(n)\log(x_{n,\vartheta}) = \xi\left(\frac{n}{\vartheta\alpha(n)}\right) + \mathcal{O}\left(\frac{\log\left(\frac{n}{\alpha(n)}+1\right)}{\alpha(n)}\right)$$

and

$$I'(\xi(u)) = \frac{\exp(\xi(u))-1}{\xi(u)} \sim \frac{\exp\left(\alpha(n)\log(x_{n,\vartheta})\right)-1}{\alpha(n)\log(x_{n,\vartheta})} = I'\left(\alpha(n)\log(x_{n,\vartheta})\right)$$

Moreover, the definition of ξ in Equation (2.6.5) entails

$$I'(\xi(u)) = u.$$

Consequently,

$$\begin{aligned}
I\left(\alpha(n)\log(x_{n,\vartheta})\right) =& I\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right) + \mathcal{O}\left(\frac{n}{\alpha(n)}\frac{\log\left(\frac{n}{\alpha(n)}+1\right)}{\alpha(n)}\right)\\
=& I\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right) + \mathcal{O}\left(\frac{n}{(\alpha(n))^2}\log\left(\frac{n}{\alpha(n)}+1\right)\right).
\end{aligned}$$

Since the error term is of lower order, it will not contribute to the asymptotic expansions in question. Recall the definition of the exponential integral Ei in Equation (2.6.20). We have

$$\begin{aligned}
I\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right) =& \int_1^{\xi\left(\frac{n}{\vartheta\alpha(n)}\right)} \frac{\exp(v)}{v}\mathrm{d}v - \int_1^{\xi\left(\frac{n}{\vartheta\alpha(n)}\right)} \frac{\mathrm{d}v}{v} + \mathcal{O}(1)\\
=& \mathrm{Ei}\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right) + \mathcal{O}\left(\log\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right)\right),
\end{aligned}$$

where the error term is of lower order. Furthermore, by Equations (2.6.21) and (2.6.5),

$$\begin{aligned}
\mathrm{Ei}\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right) &\sim \frac{\exp\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right)}{\xi\left(\frac{n}{\vartheta\alpha(n)}\right)}\sum_{j=0}^{\infty}\frac{j!}{\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right)^{j}} \\
&\sim \left(\frac{n}{\vartheta\alpha(n)}+\frac{1}{\xi\left(\frac{n}{\vartheta\alpha(n)}\right)}\right)\sum_{j=0}^{\infty}\frac{j!}{\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right)^{j}} \\
&\sim \frac{n}{\vartheta\alpha(n)}\sum_{j=0}^{\infty}\frac{j!}{\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right)^{j}}
\end{aligned}$$

holds as n tends to infinity. So we have proved

$$m_{n,\alpha} \sim \frac{n}{\alpha(n)}\sum_{j=0}^{\infty}\frac{j!}{\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right)^{j}}.$$

As to $v_{n,\alpha}$, consider

$$-n^2\left(\sum_{j=1}^{\alpha(n)} j\vartheta x_{n,\vartheta}^{j}\right)^{-1} = -\frac{n}{\alpha(n)}\sum_{j=0}^{\infty}\left(\frac{\alpha(n)}{n}\vartheta x_{n,\vartheta}+\alpha(n)(x_{n,\vartheta}-1)\right)^{-j}+\mathcal{O}(1) \tag{2.6.25}$$

by Proposition 2.6.8. A Taylor expansion yields

$$\begin{aligned}
\alpha(n)(x_{n,\vartheta}-1) =& \alpha(n)\log(x_{n,\vartheta})+\mathcal{O}\left(\alpha(n)(x_{n,\vartheta}-1)^2\right) \\
=& \xi\left(\frac{n}{\vartheta\alpha(n)}\right)+\mathcal{O}\left(\frac{\left(\xi\left(\frac{n}{\alpha(n)}\right)\right)^2}{\alpha(n)}+\frac{\log\left(\frac{n}{\alpha(n)}+1\right)}{\alpha(n)}\right),
\end{aligned} \tag{2.6.26}$$

by Lemma 2.6.9. Further note that

$$\frac{\alpha(n)}{n}\vartheta x_{n,\vartheta} = \mathcal{O}\left(\frac{\alpha(n)}{n}\right)$$

so that

$$\begin{aligned}
&\frac{\alpha(n)}{n}\vartheta x_{n,\vartheta}+\alpha(n)(x_{n,\vartheta}-1) \\
=&\xi\left(\frac{n}{\vartheta\alpha(n)}\right)+\mathcal{O}\left(\frac{\left(\xi\left(\frac{n}{\alpha(n)}\right)\right)^2}{\alpha(n)}+\frac{\log\left(\frac{n}{\alpha(n)}+1\right)}{\alpha(n)}+\frac{\alpha(n)}{n}\right)
\end{aligned}$$

holds. We therefore obtain for fixed j that

$$\begin{aligned}
&\left(\frac{\alpha(n)}{n}\vartheta x_{n,\vartheta}+\alpha(n)(x_{n,\vartheta}-1)\right)^{-j} \\
=&\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right)^{-j}+\mathcal{O}\left(\frac{\left(\xi\left(\frac{n}{\alpha(n)}\right)\right)^2}{\alpha(n)}+\frac{\log\left(\frac{n}{\alpha(n)}+1\right)}{\alpha(n)}+\frac{\alpha(n)}{n}\right).
\end{aligned} \tag{2.6.27}$$

Since the error term in Equation (2.6.27) decays faster than any power of $\log(n)$ by Equation (2.0.2), we may apply Equation (2.6.27) to Equation (2.6.25) for a finite number of indices j without affecting the asymptotic expansion. By the definition of asymptotic expansion, we then conclude

$$-n^2\left(\sum_{j=1}^{\alpha(n)} j\vartheta x_{n,\vartheta}^{j}\right)^{-1} \sim -\frac{n}{\alpha(n)}\sum_{j=0}^{\infty}\left(\xi\left(\frac{n}{\vartheta\alpha(n)}\right)\right)^{-j}$$

as $n\to\infty$. The claim follows. □

2.6.3. Further Proofs. This section collects the proofs of Propositions 2.6.6 and 2.6.8.

PROOF OF PROPOSITION 2.6.6. We have

$$\begin{aligned}\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}x_{n,\vartheta}^{j}&=\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}+\vartheta\int_{1}^{x_{n,\vartheta}}\sum_{j=1}^{\alpha(n)}w^{j-1}\mathrm{d}w\\&=\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}+\vartheta\int_{0}^{\log(x_{n,\vartheta})}\frac{\exp(\alpha(n)v)-1}{v}\frac{v\exp(v)\,\mathrm{d}v}{\exp(v)-1},\end{aligned}\tag{2.6.28}$$

where the second line applies

$$\sum_{j=1}^{\alpha(n)}w^{j-1}=\frac{w^{\alpha(n)}-1}{w-1}\tag{2.6.29}$$

and the substitution $v=\log(w)$ (cf. also [**45**]). It is a classical result which can be proved with the comparison criterion that the harmonic numbers satisfy

$$\sum_{j=1}^{\alpha(n)}\frac{1}{j}=\log(\alpha(n))+\mathcal{O}(1).\tag{2.6.30}$$

Let

$$g(v):=\frac{v\exp(v)}{\exp(v)-1}\tag{2.6.31}$$

and note that the function g occurs as a factor in the integrand in Equation (2.6.28). A short calculation yields

$$g'(v)=\frac{\exp(v)\left[\exp(v)-1-v\right]}{(\exp(v)-1)^{2}}\tag{2.6.32}$$

and

$$g''(v)=\frac{\exp(v)\left[v\exp(v)-2\exp(v)+v+2\right]}{(\exp(v)-1)^{3}}.\tag{2.6.33}$$

Moreover, an expansion of the exponential functions in the numerator of g'' shows that all coefficients in the resulting power series are non-negative. Hence,

$$g''(v)\geq 0$$

for all $v\geq 0$. Note also that g'' is bounded on $[0,\infty)$.
By expanding g about $\log(x_{n,\vartheta})$ and applying Equations (2.6.31) and (2.6.32), the integrand in Equation (2.6.28) can be calculated to satisfy

$$\begin{aligned}\frac{\exp(\alpha(n)v)-1}{v}\frac{v\exp(v)}{\exp(v)-1}=&\frac{\log(x_{n,\vartheta})}{1-x_{n,\vartheta}^{-1}}\frac{\exp(\alpha(n)v)-1}{v}\\&+\frac{(x_{n,\vartheta}-1)-\log(x_{n,\vartheta})}{(x_{n,\vartheta}-1)\left(1-x_{n,\vartheta}^{-1}\right)}\frac{\exp(\alpha(n)v)-1}{v}\left(v-\log(x_{n,\vartheta})\right)\\&+\frac{g''(\Xi(v))}{2}\frac{\exp(\alpha(n)v)-1}{v}\left(v-\log(x_{n,\vartheta})\right)^{2}\\=&\frac{1}{1-x_{n,\vartheta}^{-1}}\left[1-\frac{\log(x_{n,\vartheta})}{x_{n,\vartheta}-1}\right](\exp(\alpha(n)v)-1)\\&+\frac{(\log(x_{n,\vartheta}))^{2}}{(x_{n,\vartheta}-1)\left(1-x_{n,\vartheta}^{-1}\right)}\frac{\exp(\alpha(n)v)-1}{v}\\&+\frac{g''(\Xi(v))}{2}\frac{\exp(\alpha(n)v)-1}{v}\left(v-\log(x_{n,\vartheta})\right)^{2},\end{aligned}$$

where $0 \leq \Xi(v) \leq \log(x_{n,\vartheta})$ by Taylor's theorem. The next step is integrating each summand separately. Firstly, we have

$$\begin{aligned} I_1 :=& \int_0^{\log(x_{n,\vartheta})} \left(\exp\left(\alpha(n)v\right) - 1\right) \mathrm{d}v \\ =& \frac{x_{n,\vartheta}^{\alpha(n)}}{\alpha(n)} - \log\left(x_{n,\vartheta}\right) - \frac{1}{\alpha(n)}. \end{aligned} \tag{2.6.34}$$

Substituting $w = \frac{v}{\alpha(n)}$ yields

$$\begin{aligned} &\int_0^{\log(x_{n,\vartheta})} \frac{\exp\left(\alpha(n)v\right) - 1}{v} \mathrm{d}v \\ =& \int_0^{\alpha(n)\log(x_{n,\vartheta})} \frac{\exp(w) - 1}{w} \mathrm{d}w \\ =& \int_0^1 \frac{\exp(w) - 1}{w} \mathrm{d}w + \int_1^{\alpha(n)\log(x_{n,\vartheta})} \frac{\exp(w)}{w} \mathrm{d}w - \int_1^{\alpha(n)\log(x_{n,\vartheta})} \frac{\mathrm{d}w}{w}. \end{aligned}$$

Clearly,

$$\int_0^1 \frac{\exp(w) - 1}{w} \mathrm{d}w - \int_1^{\alpha(n)\log(x_{n,\vartheta})} \frac{\mathrm{d}w}{w} = -\log\log\left(x_{n,\vartheta}^{\alpha(n)}\right) + \mathcal{O}(1) \tag{2.6.35}$$

and

$$\begin{aligned} \int_1^{\alpha(n)\log(x_{n,\vartheta})} \frac{\exp(w)}{w} \mathrm{d}w =& \mathrm{Ei}(1) + \int_1^{\alpha(n)\log(x_{n,\vartheta})} \frac{\exp(w)}{w} \mathrm{d}w - \mathrm{Ei}(1) \\ =& \mathrm{Ei}\left(\alpha(n)\log\left(x_{n,\vartheta}\right)\right) + \mathcal{O}(1) \end{aligned} \tag{2.6.36}$$

by Equation (2.6.20). By combining Equations (2.6.35) and (2.6.36), we arrive at

$$\int_0^{\log(x_{n,\vartheta})} \frac{\exp\left(\alpha(n)v\right) - 1}{v} \mathrm{d}v = \mathrm{Ei}\left(\alpha(n)\log\left(x_{n,\vartheta}\right)\right) - \log\log\left(x_{n,\vartheta}^{\alpha(n)}\right) + \mathcal{O}(1).$$

An expansion of the exponential integral according to Equation (2.6.21) then leads to

$$\begin{aligned} I_2 :=& \int_0^{\log(x_{n,\vartheta})} \frac{\exp\left(\alpha(n)v\right) - 1}{v} \mathrm{d}v \\ =& \frac{x_{n,\vartheta}^{\alpha(n)}}{\alpha(n)\log\left(x_{n,\vartheta}\right)} \left[1 + \frac{1}{\alpha(n)\log\left(x_{n,\vartheta}\right)} + \frac{2}{\left(\alpha(n)\log\left(x_{n,\vartheta}\right)\right)^2}\right] \\ &+ \mathcal{O}\left(\frac{x_{n,\vartheta}^{\alpha(n)}}{\alpha(n)\log\left(x_{n,\vartheta}\right)} \frac{1}{\left(\alpha(n)\log\left(x_{n,\vartheta}\right)\right)^3}\right) \end{aligned} \tag{2.6.37}$$

since $\log\log\left(x_{n,\vartheta}^{\alpha(n)}\right)$ and $\mathcal{O}(1)$ are of lower order than the error term, which is a consequence of Lemma 2.6.5.
Recall that $x_{n,\vartheta} = x_{n,\alpha}\left(\frac{n}{\alpha(n)\vartheta}\right)$. Due to Equation (2.2.9),

$$x_{n,\vartheta}^{\alpha(n)} = 1 + \frac{n}{\vartheta}\left(1 - x_{n,\vartheta}^{-1}\right) = \frac{n}{\vartheta}\left(1 - x_{n,\vartheta}^{-1}\right) + \mathcal{O}(1) \sim \frac{n\left(1 - x_{n,\vartheta}^{-1}\right)}{\vartheta} \tag{2.6.38}$$

holds. If we apply Equations (2.6.34), (2.6.37), and (2.6.38) to the integral in Equation (2.6.28) and define

$$p_\alpha(n) := \vartheta \int_0^{\log(x_{n,\vartheta})} \frac{g''\left(\Xi(v)\right)}{2} \frac{\exp\left(\alpha(n)v\right) - 1}{v} \left(v - \log\left(x_{n,\vartheta}\right)\right)^2 \mathrm{d}v,$$

we obtain

$$
\begin{aligned}
&\vartheta \int_0^{\log(x_{n,\vartheta})} \frac{\exp(\alpha(n)v)-1}{v} \frac{v\exp(v)}{\exp(v)-1}\mathrm{d}v \\
=&\vartheta I_1 \frac{1}{1-x_{n,\vartheta}^{-1}}\left[1-\frac{\log(x_{n,\vartheta})}{x_{n,\vartheta}-1}\right] + \vartheta I_2 \frac{(\log(x_{n,\vartheta}))^2}{(x_{n,\vartheta}-1)\left(1-x_{n,\vartheta}^{-1}\right)} + p_\alpha(n) \\
=&\frac{n}{\alpha(n)}\left[1-\frac{\log(x_{n,\vartheta})}{x_{n,\vartheta}-1}\right] + \mathcal{O}\left(\frac{1+\log(x_{n,\vartheta})}{1-x_{n,\vartheta}^{-1}}\left(1-\frac{\log(x_{n,\vartheta})}{x_{n,\vartheta}-1}\right)\right) \\
&+\frac{n}{\alpha(n)}\left[1+\frac{1}{\alpha(n)\log(x_{n,\vartheta})}+\frac{2}{(\alpha(n)\log(x_{n,\vartheta}))^2}\right]\frac{\log(x_{n,\vartheta})}{(x_{n,\vartheta}-1)} \\
&+\mathcal{O}\left(\frac{n}{(\alpha(n))^4(x_{n,\vartheta}-1)(\log(x_{n,\vartheta}))^2}\right) + p_\alpha(n).
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&\vartheta \int_0^{\log(x_{n,\vartheta})} \frac{\exp(\alpha(n)v)-1}{v} \frac{v\exp(v)}{\exp(v)-1}\mathrm{d}v \\
=&\frac{n}{\alpha(n)} + \frac{n}{(\alpha(n))^2(x_{n,\vartheta}-1)} + 2\frac{n}{(\alpha(n))^3(x_{n,\vartheta}-1)\log(x_{n,\vartheta})} \\
&+p_\alpha(n) + \mathcal{O}\left(\frac{n}{(\alpha(n))^4(x_{n,\vartheta}-1)(\log(x_{n,\vartheta}))^2}\right) \\
&+\mathcal{O}\left(x_{n,\vartheta}\frac{1+\log(x_{n,\vartheta})}{x_{n,\vartheta}-1}\left(1-\frac{\log(x_{n,\vartheta})}{x_{n,\vartheta}-1}\right)\right).
\end{aligned}
\tag{2.6.39}
$$

In order to finish the proof, we need to control the behaviour of the second error term in Equation (2.6.39) and of $p_\alpha(n)$. Concerning the error term, there are two cases to consider: If $\alpha(n) \geq \log(n)$, then $x_{n,\vartheta} > 1$ is bounded from above for large n by Lemma 2.2.1. By Taylor's Theorem, we have

$$
\begin{aligned}
&x_{n,\vartheta}\frac{1+\log(x_{n,\vartheta})}{x_{n,\vartheta}-1}\left(1-\frac{\log(x_{n,\vartheta})}{x_{n,\vartheta}-1}\right) \\
=&\frac{1+\mathcal{O}(x_{n,\vartheta}-1)}{x_{n,\vartheta}-1}\left(1-\frac{x_{n,\vartheta}-1+\mathcal{O}\left((x_{n,\vartheta}-1)^2\right)}{x_{n,\vartheta}-1}\right) \\
=&\mathcal{O}(1).
\end{aligned}
$$

In order to verify

$$
\mathcal{O}(1) \subset \mathcal{O}\left(\frac{n}{(\alpha(n))^4(x_{n,\vartheta}-1)(\log(x_{n,\vartheta}))^2}\right) \tag{2.6.40}
$$

in this case, we require a suitable upper bound for $\alpha(n)(x_{n,\vartheta}-1)$. By Lemma 2.6.5, we have

$$
\alpha(n)(x_{n,\vartheta}-1) \approx \log\left(\frac{n}{\alpha(n)}\right)
$$

for large n, and Equation (2.6.40) follows. Consider the second case of $3 \leq \alpha(n) < \log(n)$. By Lemma 2.2.1, $x_{n,\vartheta} > c$ for some $c > 1$ for large n, so

$$
x_{n,\vartheta}\frac{1+\log(x_{n,\vartheta})}{x_{n,\vartheta}-1}\left(1-\frac{\log(x_{n,\vartheta})}{x_{n,\vartheta}-1}\right) = \mathcal{O}(x_{n,\vartheta}),
$$

and we have to show that

$$
x_{n,\vartheta} = \mathcal{O}\left(\frac{n}{(\alpha(n))^4(x_{n,\vartheta}-1)(\log(x_{n,\vartheta}))^2}\right) \tag{2.6.41}
$$

in this case. By Equation (2.2.5), we have

$$\begin{aligned}\alpha(n)(x_{n,\vartheta}-1) \leq & \alpha(n) x_{n,\vartheta} \\ \leq & \alpha(n)\left(\frac{n}{\vartheta}\right)^{\frac{1}{\alpha(n)}},\end{aligned}$$

which entails Equation (2.6.41).
Note further that Equations (2.6.40) and (2.6.41) imply

$$\mathcal{O}(1) \subset \mathcal{O}\left(\frac{n}{(\alpha(n))^4 (x_{n,\vartheta}-1)(\log(x_{n,\vartheta}))^2}\right)$$

for all α, which we apply to the error term in Equation (2.6.30).
The function $p_\alpha(n)$ is non-negative by definition since $g''(v) \geq 0$ for all $v \geq 0$. By partial integration, we obtain

$$\begin{aligned} I_3 := & \int_0^{\log(x_{n,\vartheta})} (\exp(\alpha(n) v) - 1) v \mathrm{d}v \\ = & \frac{x_{n,\vartheta}^{\alpha(n)}}{(\alpha(n))^2}[\alpha(n)\log(x_{n,\vartheta}) - 1] + \frac{1}{(\alpha(n))^2} - \frac{(\log(x_{n,\vartheta}))^2}{2}. \end{aligned} \tag{2.6.42}$$

Since g'' is bounded,

$$\begin{aligned} p_\alpha(n) = & \mathcal{O}\left(I_3 - 2I_1\log(x_{n,\vartheta}) + I_2(\log(x_{n,\vartheta}))^2\right) \\ \subset & \mathcal{O}\left(\frac{x_{n,\vartheta}^{\alpha(n)}}{\alpha(n)}\left[\log(x_{n,\vartheta}) - \frac{1}{\alpha(n)} - 2\log(x_{n,\vartheta}) + \log(x_{n,\vartheta}) + \frac{1}{\alpha(n)} + \frac{2}{(\alpha(n))^2\log(x_{n,\vartheta})}\right]\right) \\ \subset & \mathcal{O}\left(\frac{x_{n,\vartheta}^{\alpha(n)}}{(\alpha(n))^3\log(x_{n,\vartheta})}\right) \end{aligned}$$

follows from Equations (2.6.34), (2.6.37), and (2.6.42). The claim is proved. □

Proof of Proposition 2.6.8. Recall that

$$-n^2\left(\sum_{j=1}^{\alpha(n)} j x_{n,\vartheta}^j\right)^{-1} = n\gamma(n)\frac{x'_{n,C}(0)}{x_{n,\vartheta}}$$

by Equation (2.6.14). Applying the formula for the geometric series to Equation (2.6.9) yields

$$\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}\left[(x_{n,C}(s))^{\alpha(n)} - 1\right] = n\left(1 - \frac{1}{x_{n,C}(s)}\right). \tag{2.6.43}$$

If we differentiate Equation (2.6.43) with respect to s, we obtain

$$\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{\gamma(n)}\left[(x_{n,C}(s))^{\alpha(n)} - 1\right] + \alpha(n)\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}(x_{n,C}(s))^{\alpha(n)}\frac{x'_{n,C}(s)}{x_{n,C}(s)} = \frac{n}{x_{n,C}(s)}\frac{x'_{n,C}(s)}{x_{n,C}(s)}.$$

Solving for $x'_{n,C}(s)/x_{n,C}(s)$ then leads to

$$\frac{x'_{n,C}(s)}{x_{n,C}(s)} = \frac{1}{\gamma(n)}\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}\left[(x_{n,C}(s))^{\alpha(n)} - 1\right]}{n/x_{n,C}(s) - \alpha(n)\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}(x_{n,C}(s))^{\alpha(n)}}.$$

By Equation (2.6.43),

$$\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}(x_{n,C}(s))^{\alpha(n)} = \vartheta \mathrm{e}^{\frac{s}{\gamma(n)}} + n\frac{x_{n,C}(s) - 1}{x_{n,C}(s)},$$

so

$$\begin{aligned}
\frac{x_{n,C}'(s)}{x_{n,C}(s)} =& \frac{1}{\gamma(n)} \frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}} + n\frac{x_{n,C}(s)-1}{x_{n,C}(s)} - \vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{n/x_{n,C}(s) - \alpha(n)\left(\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}} + n\frac{x_{n,C}(s)-1}{x_{n,C}(s)}\right)}\\
=& \frac{1}{\gamma(n)} \frac{x_{n,C}(s) - 1}{1 - \frac{\alpha(n)}{n} x_{n,C}(s)\, \vartheta \mathrm{e}^{\frac{s}{\gamma(n)}} - \alpha(n)\left(x_{n,C}(s) - 1\right)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
n\gamma(n)\frac{x_{n,C}'(0)}{x_{n,\vartheta}} =& \frac{n\left(x_{n,\vartheta}-1\right)}{1 - \frac{\alpha(n)}{n}\vartheta x_{n,\vartheta} - \alpha(n)\left(x_{n,\vartheta}-1\right)}\\
=& \frac{n}{\alpha(n)} \frac{\frac{\alpha(n)}{n}\vartheta x_{n,\vartheta} + \alpha(n)\left(x_{n,\vartheta}-1\right) - \frac{\alpha(n)}{n}\vartheta x_{n,\vartheta}}{1 - \frac{\alpha(n)}{n}\vartheta x_{n,\vartheta} - \alpha(n)\left(x_{n,\vartheta}-1\right)}\\
=& -\frac{n}{\alpha(n)} \frac{1}{1 - \frac{1}{\frac{\alpha(n)}{n}\vartheta x_{n,\vartheta} + \alpha(n)\left(x_{n,\vartheta}-1\right)}} + \mathcal{O}(1)\\
=& -\frac{n}{\alpha(n)} \sum_{j=0}^{\infty} \left(\frac{\alpha(n)}{n}\vartheta x_{n,\vartheta} + \alpha(n)\left(x_{n,\vartheta}-1\right)\right)^{-j} + \mathcal{O}(1)
\end{aligned}$$

follows. Here, the third line applies $\alpha(n)\left(x_{n,\vartheta}-1\right) \geq \alpha(n)\log\left(x_{n,\vartheta}\right) \to \infty$ by Lemma 2.6.5 and the fourth line holds due to the formula for the geometric series. This proves Equation (2.6.22). Moreover, a short calculation yields

$$\begin{aligned}
&\frac{1}{\frac{\alpha(n)}{n}\vartheta x_{n,\vartheta} + \alpha(n)\left(x_{n,\vartheta}-1\right)}\\
=& \frac{1}{\alpha(n)\left(x_{n,\vartheta}-1\right)} - \frac{\alpha(n)}{n} \frac{\vartheta x_{n,\vartheta}}{\frac{(\alpha(n))^2}{n}\vartheta x_{n,\vartheta}\left(x_{n,\vartheta}-1\right) + (\alpha(n))^2\left(x_{n,\vartheta}-1\right)^2}
\end{aligned}$$

and the saddle point may be bounded from above by

$$x_{n,\vartheta} \leq \left(\frac{n}{\alpha(n)\vartheta}\right)^{\frac{2}{1+\alpha(n)}}$$

due to Equation (2.2.7). Hence,

$$\begin{aligned}
\frac{\alpha(n)}{n} \frac{\vartheta x_{n,\vartheta}}{\frac{(\alpha(n))^2}{n}\vartheta x_{n,\vartheta}\left(x_{n,\vartheta}-1\right) + (\alpha(n))^2\left(x_{n,\vartheta}-1\right)^2} \leq& \frac{\alpha(n)}{n} \frac{\vartheta x_{n,\vartheta}}{(\alpha(n))^2\left(x_{n,\vartheta}-1\right)^2}\\
=& o\left(\left(\frac{1}{\alpha(n)\left(x_{n,\vartheta}-1\right)}\right)^2\right)
\end{aligned}$$

when $\alpha(n) \geq 2$, which proves Equation (2.6.23). □

2.7. Cumulative Cycle Numbers and Cumulative Index Numbers

This section deals with the random variables

$$K_b := \sum_{j=1}^{b} C_j \text{ and } S_b := \sum_{j=1}^{b} jC_j$$

to which we will refer as cumulative cycle numbers and cumulative index numbers, respectively. Intuitively, K_b is the number of cycles of lengths less than or equal to b, while S_b is the number of indices contained in such cycles. We will prove for both K_b and S_b that there exists a limit shape and that the rescaled fluctuations about this limit shape satisfy a functional limit theorem. Moreover, limit shapes and fluctuations of K_b and S_b will live on the same scale of b.

Although different in nature, the statements in this section are the counterparts to, on the one hand, convergence of the lengths of the longest cycles to the Poisson-Dirichlet distribution (see Section 1.1.3) and, on the other hand, the functional limit theorem for cumulative cycle numbers (cf. Section 1.1.2.3) in the classical case, which live on different scales. So, under the Ewens measure, there are two manifestly distinct concepts of typical cycles which, however, coincide asymptotically for random permutations without macroscopic cycles (see Section 2.9). As we will see in Section 2.8, this entails that the longest cycles in constrained random permutations occur on a different scale.

The results and most proofs presented in this section have been presented in [**10**] for $\vartheta = 1$. This thesis additionally gives a proof of Theorem 2.7.6 and substantially expands the proofs of tightness. Moreover, the proofs of certain auxiliary lemmata have been simplified.

Let α be as in Equation (2.0.2) throughout this section.

2.7.1. Results. Concerning the limit shapes of cumulative cycle and index numbers, everything will fall into place once we find the regime of b such that K_b and S_b are asymptotically a fraction of the total number of cycles or indices, respectively. By Theorem 2.6.3, we know that, under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$,

$$\lim_{n\to\infty} \frac{C}{\frac{n}{\alpha(n)}} = \lim_{n\to\infty} \frac{K_{\alpha(n)}}{\frac{n}{\alpha(n)}} = 1, \tag{2.7.1}$$

e.g. in distribution. By definition, the total number of indices satisfies

$$S_{\alpha(n)} = n. \tag{2.7.2}$$

We are looking for b such that, for $t \in [0,1]$, we have

$$K_b \sim t\frac{n}{\alpha(n)}$$

and

$$S_b \sim tn.$$

As Theorems 2.7.1 and 2.7.2 will state, the right regime for both cases is given by

$$b_t(n) := \max\left\{\left\lfloor \alpha(n) + \log(t)\,\frac{\alpha(n)}{\log\left(\frac{n}{\alpha(n)}\right)}\right\rfloor, 0\right\} \tag{2.7.3}$$

for $t \in [0,1]$. Note that $b_1(n) = \alpha(n)$, $b_0(n) = 0$, and that $b_t(n) \sim \alpha(n)$ pointwise in $t \in (0,1]$. The random functions $t \mapsto K_{b_t(n)}$ and $t \mapsto S_{b_t(n)}$ have the following limit shapes.

Theorem 2.7.1 ([**10**, Theorem 2.5]). *For each $\epsilon > 0$, we have*

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\sup_{t\in[0,1]}\left|\frac{K_{b_t(n)}}{n/\alpha(n)} - t\right| > \epsilon\right] \to 0$$

as $n \to \infty$.

Theorem 2.7.2 ([**10**, Remark 3, p.5]). *For each $\epsilon > 0$, we have*

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\sup_{t\in[0,1]}\left|\frac{S_{b_t(n)}}{n} - t\right| > \epsilon\right] \to 0$$

as $n \to \infty$.

Remark 2.7.3. Theorems 2.7.1 and 2.7.2 above show that the cumulative cycle and index numbers move in lockstep in the regime of $b_t(n)$ with the appropriate scalings. This insight will be utilized in the proof of Theorem 2.7.2 which applies the statements about cumulative cycle numbers.

Remark 2.7.4. By looking at the case $t = 1$, the theorems yield the statements in Equations (2.7.1) and (2.7.2). If we consider the limit $t \to 0$, we see that no positive fractions of the total cycle count and the total number of indices occur below the regime given by $b_t(n)$.

We now determine the fluctuations about the limit shapes by centering and suitably rescaling the random functions in question.

Theorem 2.7.5 ([**10**, Theorem 2.6]). *Let*

$$L_t(n) := \frac{K_{b_t(n)} - \sum_{j=1}^{b_t(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j}{\sqrt{n/\alpha(n)}}$$

for $t \in [0,1]$. *Then* $(L_t(n))_{t\in[0,1]}$ *under* $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ *converges in distribution to the Brownian bridge in* $\mathcal{D}[0,1]$ *as* $n \to \infty$. *Here,* $\mathcal{D}[0,1]$ *is the set of càd-làg functions on* $[0,1]$ *endowed with the Skorohod topology.*

Theorem 2.7.6 ([**10**, Remark 2, p.6]). *Let*

$$\tilde{L}_t(n) := \frac{S_{b_t(n)} - \sum_{j=1}^{b_t(n)} \vartheta x_{n,\vartheta}^j}{\sqrt{n\alpha(n)}}$$

for $t \in [0,1]$. *Then* $\left(\tilde{L}_t(n)\right)_{t\in[0,1]}$ *under* $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ *converges in distribution to the Brownian bridge in* $\mathcal{D}[0,1]$ *as* $n \to \infty$.

Remark 2.7.7. Note that we do not subtract the expected value in the definition of $L_t(n)$, but the term $\sum_{j=1}^{b_t(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j$ which is only asymptotically equivalent (pointwise in t) to the expected value $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[K_{b_t(n)}\right]$. Yet, once we have proved Theorem 2.7.5, we know that $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[|L_t(n)|^2\right]$ is bounded for large n and even uniformly in t. Hence, by uniform integrability, we have $\lim_{n\to\infty} \mathbb{E}_{n,\alpha}^{(\vartheta)}[L_t(n)] = 0$, i.e. $\frac{\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[K_{b_t(n)}\right] - \sum_{j=1}^{b_t(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j}{\sqrt{n/\alpha(n)}} \to 0$ for each t. So we may substitute $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[K_{b_t(n)}\right]$ for $\sum_{j=1}^{b_t(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j$ in the definition of $L_t(n)$ and retain convergence of the finite-dimensional distributions. The question of convergence as a stochastic process is more involved, but we still expect it to hold. An analogous statement applies to $\tilde{L}_t(n)$.

Remark 2.7.8. Even though cumulative cycle and index numbers still move exactly in parallel, other than Theorem 2.7.2, we cannot trace back Theorem 2.7.6 to the case of cumulative cycle numbers. So we have to prove it in a way analogous to the proof of Theorem 2.7.5.

Remark 2.7.9. If we consider the marginal at $t = 1$, Theorem 2.7.5 yields that the variance of the total number of cycles $C = K_{\alpha(n)}$ rescaled by a factor of $\sqrt{\frac{n}{\alpha(n)}}$ tends to 0 in the limit of $n \to \infty$. Recall, however, that by Theorem 2.6.3 there is a different rescaling, namely $\sqrt{\frac{n}{\alpha(n)(\log(n/\alpha(n)))^2}}$, such that we have weak convergence to the standard normal distribution. Since $S_{\alpha(n)} = n$ is deterministic, an analogous statement cannot be true in this case.
As the limit is the Brownian bridge, the fluctuations are asymptotically 0 at the boundaries $t = 0$ and $t = 1$ and maximal in the centre of the interval when $t = \frac{1}{2}$.

Remark 2.7.10. The fact that Theorems 2.7.5 and 2.7.6 state convergence to the Brownian bridge is due to the choice of t-dependence in Equation (2.7.3). There are other possibilities aside from the logarithmic parametrization, an alternative being the (reversed) linear parametrization given by

$$\tilde{b}_\tau(n) := \max\left\{\alpha(n) - \left\lfloor \tau \frac{\alpha(n)}{\log(n/\alpha(n))} \right\rfloor, 0\right\}$$

for $\tau \in [0,\infty)$. This definition then implies an exponential limit shape, i.e.

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\sup_{\tau\in[0,\infty)} \left|\frac{K_{\tilde{b}_\tau(n)}}{n/\alpha(n)} - \mathrm{e}^{-\tau}\right| > \epsilon\right] \to 0,$$

and weak convergence of the fluctuations to a Gaussian process with covariance $\mathrm{e}^{-\tau_1}(1-\mathrm{e}^{-\tau_2})$ for $\tau_1 \geq \tau_2$. Note, however, that this functional limit theorem is substantially weaker since it does not include the point $\tau = \infty$ (which corresponds to $t = 0$ in the logarithmic parametrization) and intuitively stretches the half-open unit interval. A glance at Equation (1.2.3) and Proposition 1.2.9 shows how this entails discounting the contribution “at infinity”. Indeed, considerably more extended calculations and a more sophisticated ansatz are required to verify tightness in the case of the logarithmic parametrization (cf., in particular, Lemma A.1.8 and Remark 2.7.16).
If one additionally rescales $K_{\tilde{b}_\tau(n)}$ exponentially by a factor of $\mathrm{e}^{\frac{\tau}{2}}$ and considers

$$\bar{K}_\tau(n) := \mathrm{e}^{\frac{\tau}{2}} K_{\tilde{b}_\tau(n)},$$

one arrives at the exponential limit shape $\mathrm{e}^{-\frac{\tau}{2}}$. In this case, the fluctuations about the limit shape converge to a standard Ornstein-Uhlenbeck process with covariance $\mathrm{e}^{-|\tau_1-\tau_2|/2} - \mathrm{e}^{-|\tau_1+\tau_2|/2}$. In particular, the variance of the Ornstein-Uhlenbeck process converges as $\tau \to \infty$ (which corresponds to the regime of small cycle lengths). See also Remark 5 after Theorem 2.6 in [**10**].

Remark 2.7.11. The following Figures 2.7.1 and 2.7.2 show the rescaled cumulative cycle and index numbers as well as the whole cycle structure of a typical simulated realization of a random permutation without macroscopic cycles for $n = 10^6$, $\alpha(n) = \sqrt{n}$, and $\vartheta = 1$ (cf. Section A.2).

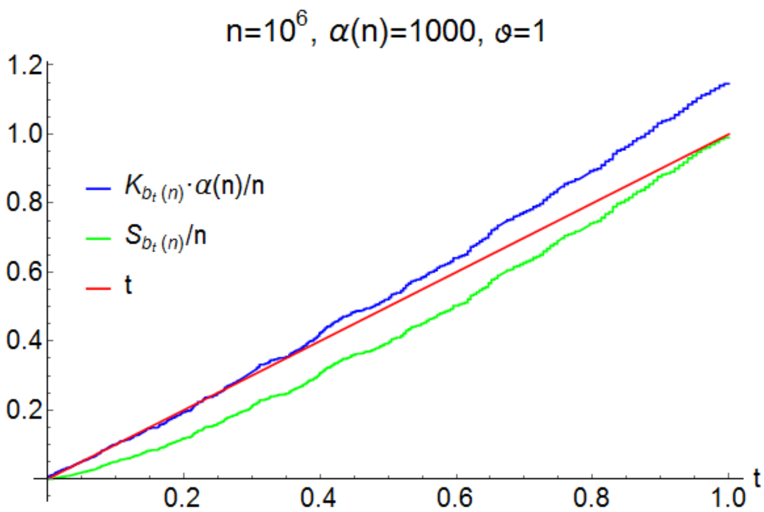

Figure 2.7.1. Rescaled cumulative cycle and index numbers of a simulated realization as well as the limit shape

In Figure 2.7.1, we apply the rescalings in Theorems 2.7.1 and 2.7.2 to $K_{b_t(n)}$ and $S_{b_t(n)}$ and further depict the limit shape. In particular, one sees that the total number of cycles $K_{b_1(n)}$ exceeds $\frac{n}{\alpha(n)}$. The reason is that, when $n = 10^6$, the contribution of the second term in the asymptotic expansion of $m_{n,\alpha}$ in Equation (2.6.6) is not negligible: If we include it, we obtain $m_{n,\alpha} \approx 10^3 + \frac{10^3}{\log(10^3)} \approx 1145$, which is a far better approximation. So the deviation from 1 can be attributed to a finite-size effect. Furthermore, one notices that the bulk of the small cycles which enter into $K_{b_t(n)}$ for small t are not yet of lengths of order $\alpha(n)$. Hence, while $K_{b_t(n)}\frac{\alpha(n)}{n}$ is close to the limit shape for small t, $S_{b_t(n)}/n$ does not increase fast enough. The resulting gap is compensated for by a greater number of cycles of lengths larger than about $b_{0.5}(n)$ to obtain $S_{b_1(n)}/n = 1$.

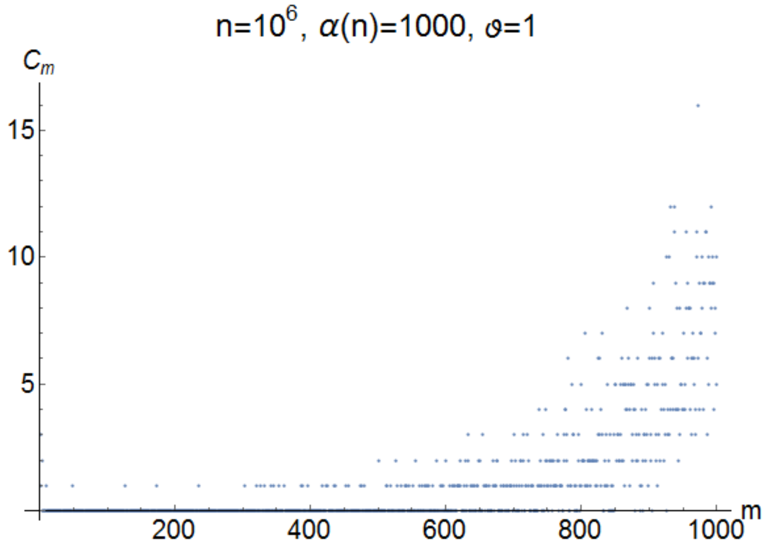

FIGURE 2.7.2. Cycle structure of a simulated realization

Figure 2.7.2 depicts the cycle structure of the realization. Even though α is in the regime in which $\mu_{\alpha(n)}(n)$ diverges, all cycle counts are less than or equal to 16. This fact can again be retraced to $n = 10^6$ not being large enough, since $\mu_{\alpha(n)}(n) = \log\left(\frac{n}{\alpha(n)}\right)$ in this case, which diverges only slowly. Indeed, here we have $\mu_{1000}\left(10^6\right) \approx 7$. Note that, in the terminology of Section 2.3.2, $m_{min}\left(10^6\right) \approx 145$, which indicates for which cycle lengths m the smallest cycle counts occur. This value is congruent with Figure 2.7.2. All in all we draw the conclusion that the convergence in Theorems 2.7.1 and 2.7.2 is quite slow.

2.7.2. Proofs of Theorems 2.7.1 and 2.7.5. In order to prove the theorems, we will first consider the finite-dimensional distributions of $\left(K_{b_t(n)}\right)_{t\in[0,1]}$. The saddle-point method is here again the instrument of choice and the approach is similar to the one adopted in Section 2.6. Let $\gamma = (\gamma(n))_{n\in\mathbb{N}}$ be a sequence which diverges as n tends to infinity, fix a natural number $M > 0$, $\boldsymbol{s} = (s_k)_{k=1}^M \in [0,\infty)^K$ and $\boldsymbol{t} = (t_k)_{k=1}^M$ with $0 = t_0 \le t_1 < t_2 \cdots < t_M \le t_{M+1} = 1$. Then, by Equation (1.3.6), we have the moment-generating function

$$\begin{aligned} M_{n,K}(\boldsymbol{s}) :=& \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\prod_{k=1}^M \exp\left(\frac{s_k}{\gamma(n)} K_{b_{t_k}(n)}\right)\right] \\ =& \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\prod_{k=1}^M \exp\left(\frac{s_k}{\gamma(n)} \sum_{j=1}^{b_{t_k}(n)} C_j\right)\right] \\ =& \frac{1}{Z_{n,\alpha,\vartheta}}\left[z^n\right]\exp\left(\sum_{k=0}^M \mathrm{e}^{\sum_{i=k+1}^M \frac{s_i}{\gamma(n)}} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \frac{\vartheta}{j} z^j\right). \end{aligned} \tag{2.7.4}$$

We want to apply Proposition 2.1.4 to Equation (2.7.4). The relevant perturbations are given by $f_n(z) = 1$, which is trivially admissible. The triangular array $\boldsymbol{q}$ is given by

$$q_{j,n} = \vartheta \mathrm{e}^{\sum_{i=k(j,n)}^M \frac{s_i}{\gamma(n)}},$$

where $k(j,n) := \min\{1 \le i \le M : b_{t_i}(n) \ge j\}$. Intuitively, this means that any index i with $b_{t_i}(n) \ge j$ contributes a factor of $\mathrm{e}^{s_i/\gamma(n)}$ to $q_{j,n}$ since $K_{b_{t_i}(n)}$ includes the number of cycles of length j in this case. The saddle point $x_{n,K}(\boldsymbol{s}) := x_{n,\boldsymbol{q}}$ is the unique positive solution of

$$n = \sum_{k=0}^M \mathrm{e}^{\sum_{i=k+1}^M \frac{s_i}{\gamma(n)}} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta\left(x_{n,K}(\boldsymbol{s})\right)^j. \tag{2.7.5}$$

Note that $x_{n,K}(\mathbf{0}) = x_{n,\vartheta}$. The next step is checking admissibility for the array $\boldsymbol{q}$, which is the content of Lemmata 2.7.12 and 2.7.13. The lemmata provide more information than is strictly necessary for the purpose at hand, but the results will be referred to in later stages of the proofs of the theorems.

LEMMA 2.7.12 ([**10**, Lemma 4.5]). *Let $\gamma(n) = \Omega(\log(n))$, $\boldsymbol{t} = (t_k)_{k=1}^M$ such that $0 = t_0 \le t_1 < t_2 < \cdots < t_M \le t_{M+1} = 1$, and $\boldsymbol{s} = (s_k)_{k=1}^M$ such that $s_k \ge 0$ for all k. Then,*

$$\alpha(n)\log(x_{n,K}(\boldsymbol{s})) \sim \log\left(\frac{n}{\alpha(n)}\right) \tag{2.7.6}$$

locally uniformly in $\boldsymbol{s}$ as $n \to \infty$. In particular,

$$\lim_{n\to\infty} x_{n,K}(\boldsymbol{s}) = 1 \tag{2.7.7}$$

locally uniformly in $\boldsymbol{s}$.

PROOF. Recall that $x_{n,\alpha}(u)$ is the unique positive solution of

$$\alpha(n)\,u = \sum_{j=1}^{\alpha(n)} (x_{n,\alpha}(u))^j$$

by Equation (2.2.1). Since $s_k \ge 0$ for all k, a comparison of the respective definitions leads to

$$x_{n,\alpha}\left(\mathrm{e}^{-\sum_{k=1}^M \frac{s_k}{\gamma(n)}} \frac{n}{\vartheta\alpha(n)}\right) \le x_{n,K}(\boldsymbol{s}) \le x_{n,\alpha}\left(\frac{n}{\vartheta\alpha(n)}\right). \tag{2.7.8}$$

By Lemma 2.2.1 and due to $\gamma(n) \to \infty$, we conclude

$$\alpha(n)\log\left(x_{n,\alpha}\left(\mathrm{e}^{-\sum_{k=1}^M \frac{s_k}{\gamma(n)}} \frac{n}{\vartheta\alpha(n)}\right)\right) \sim \log\left(\mathrm{e}^{-\sum_{k=1}^M \frac{s_k}{\gamma(n)}} \frac{n}{\vartheta\alpha(n)}\right) \sim \log\left(\frac{n}{\alpha(n)}\right)$$

locally uniformly in $\boldsymbol{s}$ and

$$\alpha(n)\log\left(x_{n,\alpha}\left(\frac{n}{\vartheta\alpha(n)}\right)\right) \sim \log\left(\frac{n}{\alpha(n)}\right).$$

Equation (2.7.6) then follows from the monotonicity property of the logarithm function, which in turn entails Equation (2.7.7). □

LEMMA 2.7.13 ([**10**, Lemma 4.6]). *Under the assumptions of Lemma 2.7.12, we have*

$$\lambda_{2,n} = \sum_{k=0}^{M} \mathrm{e}^{\sum_{i=k+1}^M \frac{s_i}{\gamma(n)}} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta j\,(x_{n,K}(\boldsymbol{s}))^j = n\alpha(n) + \mathcal{O}\left(\frac{n\alpha(n)}{\log(n)}\right)$$

locally uniformly in $\boldsymbol{s} \in [0,\infty)^M$.

PROOF. By applying Equation (2.7.8), we obtain

$$\vartheta\sum_{j=1}^{\alpha(n)} j\left(x_{n,\alpha}\left(\mathrm{e}^{-\sum_{k=1}^M \frac{s_k}{\gamma(n)}} \frac{n}{\vartheta\alpha(n)}\right)\right)^j \le \lambda_{2,n} \le \vartheta\mathrm{e}^{\sum_{k=1}^M \frac{s_k}{\gamma(n)}} \sum_{j=1}^{\alpha(n)} j\left(x_{n,\alpha}\left(\frac{n}{\vartheta\alpha(n)}\right)\right)^j$$

since $s_k \ge 0$ for all k. Lemma 2.2.1 and Equation (2.0.2) tell us that both

$$\begin{aligned}
&\vartheta\sum_{j=1}^{\alpha(n)} j\left(x_{n,\alpha}\left(\mathrm{e}^{-\sum_{k=1}^M \frac{s_k}{\gamma(n)}} \frac{n}{\vartheta\alpha(n)}\right)\right)^j \\
=&n\alpha(n)\,\mathrm{e}^{-\sum_{k=1}^M \frac{s_k}{\gamma(n)}}\left(1 + \mathcal{O}\left(\frac{1}{\log\left(\mathrm{e}^{-\sum_{k=1}^M \frac{s_k}{\gamma(n)}} \frac{n}{\vartheta\alpha(n)}\right)}\right)\right) \\
=&n\alpha(n) + \mathcal{O}\left(\frac{n\alpha(n)}{\log(n)}\right)
\end{aligned}$$

(by $\gamma(n) = \Omega(\log(n))$) and

$$\vartheta \mathrm{e}^{\sum_{k=1}^{M} \frac{s_k}{\gamma(n)}} \sum_{j=1}^{\alpha(n)} j \left(x_{n,\alpha} \left(\frac{n}{\vartheta \alpha(n)} \right) \right)^j = n\alpha(n) + \mathcal{O}\left(\frac{n\alpha(n)}{\log(n)} \right)$$

hold locally uniformly in $\boldsymbol{s}$. The claim is therefore proved. □

Note that one could weaken the assumptions about γ, but $\gamma(n) = \Omega(\log(n))$ is not too restrictive with respect to the theorems we aim to prove and one would otherwise lose the precise error term. Lemmata 2.7.12 and 2.7.13 entail that the array $\boldsymbol{q}$ satisfies conditions (1) and (2) in Definition 2.1.1 if $\gamma(n) = \Omega(\log(n))$. Since condition (3) follows from $s_k \geq 0$ for all k, we conclude that $\boldsymbol{q}$ is admissible. By applying Proposition 2.1.4 and Lemma 2.7.13 to Equation (2.7.4), we arrive at

$$M_{n,K}(\boldsymbol{s}) = \frac{1}{Z_{n,\alpha,\vartheta}} \frac{1}{\sqrt{2\pi n\alpha(n)}} \exp(h_{n,K}(\boldsymbol{s}))(1 + o(1))$$

pointwise in $\boldsymbol{s} \in [0,\infty)^M$, where

$$h_{n,K}(\boldsymbol{s}) := \sum_{k=0}^{M} \mathrm{e}^{\sum_{i=k+1}^{M} \frac{s_i}{\gamma(n)}} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \frac{\vartheta}{j} (x_{n,K}(\boldsymbol{s}))^j - n\log(x_{n,K}(\boldsymbol{s})).$$

Note that $M_{n,K}(\mathbf{0}) = 1$, so we infer

$$Z_{n,\alpha,\vartheta} \sim \frac{1}{\sqrt{2\pi n\alpha(n)}} \exp(h_{n,K}(\mathbf{0})). \tag{2.7.9}$$

In order to prove the theorems, we are going to expand the functions $h_{n,K}$ about $\boldsymbol{s} = \mathbf{0}$. The following Lemma 2.7.14 provides tools necessary for understanding the asymptotics of the derivatives of $h_{n,K}$ (and $h_{n,S}$ in Section 2.7.3 below).

Lemma 2.7.14 ([**10**, Lemma 4.8]). *We have*

$$\sum_{j=b_{t_1}(n)+1}^{b_{t_2}(n)} \vartheta x_{n,\vartheta}^j \sim (t_2 - t_1)\, n, \tag{2.7.10}$$

$$\sum_{j=b_{t_1}(n)+1}^{b_{t_2}(n)} \vartheta j x_{n,\vartheta}^j \sim (t_2 - t_1)\, n\alpha(n), \text{ and} \tag{2.7.11}$$

$$\sum_{j=b_{t_1}(n)+1}^{b_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \sim (t_2 - t_1) \frac{n}{\alpha(n)} \tag{2.7.12}$$

pointwise in $0 \leq t_1 < t_2 \leq 1$ *as* $n \to \infty$.

Proof. Recall that we have

$$\sum_{j=1}^{\alpha(n)} \vartheta x_{n,\vartheta}^j = n$$

by Equation (2.3.2). Let $t > 0$. We will prove that

$$\sum_{j=1}^{b_t(n)} \vartheta x_{n,\vartheta}^j \sim tn \tag{2.7.13}$$

pointwise in t, Equation (2.7.10) is then a direct consequence. Due to $x_{n,\vartheta} > 1$,

$$\vartheta \int_0^{b_t(n)} x_{n,\vartheta}^v \mathrm{d}v \leq \sum_{j=1}^{b_t(n)} \vartheta x_{n,\vartheta}^j \leq \vartheta \int_1^{b_t(n)+1} x_{n,\vartheta}^v \mathrm{d}v = x_{n,\vartheta} \vartheta \int_0^{b_t(n)} x_{n,\vartheta}^v \mathrm{d}v$$

follows. So, by Lemma 2.3.1, we have to show that

$$\vartheta \int_0^{b_t(n)} x_{n,\vartheta}^v \mathrm{d}v = \vartheta \frac{x_{n,\vartheta}^{b_t(n)} - 1}{\log(x_{n,\vartheta})} \sim tn.$$

Lemma 2.3.1 entails both

$$x_{n,\vartheta}^{\alpha(n)} = \frac{n}{\alpha(n)\vartheta} \log\left(\frac{n}{\alpha(n)\vartheta}\right)(1+o(1))$$

and

$$\frac{1}{\log(x_{n,\vartheta})} \sim \frac{\alpha(n)}{\log\left(\frac{n}{\alpha(n)}\right)}.$$

Note that, since $t>0$ has been fixed,

$$0 < \frac{b_t(n)}{\alpha(n)} \leq 1$$

for n large enough. Hence,

$$\begin{aligned}
x_{n,\vartheta}^{b_t(n)} &= \left[x_{n,\vartheta}^{\alpha(n)}\right]^{\frac{b_t(n)}{\alpha(n)}} \\
&= \exp\left[\frac{b_t(n)}{\alpha(n)} \log\left(\frac{n}{\alpha(n)\vartheta}\log\left(\frac{n}{\alpha(n)\vartheta}\right)\right)\right](1+o(1)) \\
&= \frac{n}{\alpha(n)\vartheta}\log\left(\frac{n}{\alpha(n)\vartheta}\right)\exp\left[\log(t)\frac{\log\left(\frac{n}{\alpha(n)\vartheta}\log\left(\frac{n}{\alpha(n)\vartheta}\right)\right)}{\log\left(\frac{n}{\alpha(n)}\right)}\right](1+o(1)) \\
&\sim t\frac{n}{\alpha(n)\vartheta}\log\left(\frac{n}{\alpha(n)}\right)
\end{aligned}$$

pointwise in t, where the third line applies the definiton of $b_t(n)$ in Equation (2.7.3). So Equation (2.7.13) is proved and Equation (2.7.10) follows.
Since

$$\sum_{j=1}^{\alpha(n)} \vartheta j x_{n,\vartheta}^j \sim n\alpha(n) \tag{2.7.14}$$

by Lemma 2.3.1, Equation (2.7.11) is a consequence of

$$\sum_{j=1}^{b_t(n)} \vartheta j x_{n,\vartheta}^j \sim tn\alpha(n) \tag{2.7.15}$$

pointwise in t. Note that

$$\sum_{j=1}^{\alpha(n)} \vartheta j x_{n,\vartheta}^j - \alpha(n)\sum_{j=b_t(n)+1}^{\alpha(n)} \vartheta x_{n,\vartheta}^j \leq \sum_{j=1}^{b_t(n)} \vartheta j x_{n,\vartheta}^j \leq \alpha(n)\sum_{j=1}^{b_t(n)} \vartheta x_{n,\vartheta}^j.$$

So Equation (2.7.15) holds pointwise in t due to Equation (2.7.10), and Equation (2.7.11) is proved. A similar approach is also suitable for proving Equation (2.7.12). Proposition 2.6.3 yields

$$\sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \sim \frac{n}{\alpha(n)}. \tag{2.7.16}$$

Since we have

$$\frac{1}{\alpha(n)}\sum_{j=1}^{b_t(n)} \vartheta x_{n,\vartheta}^j \leq \sum_{j=1}^{b_t(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \leq \sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j - \frac{1}{\alpha(n)}\sum_{j=b_t(n)+1}^{\alpha(n)} \vartheta x_{n,\vartheta}^j,$$

Equation (2.7.10) entails

$$\sum_{j=1}^{b_t(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \sim t\frac{n}{\alpha(n)}$$

pointwise in t. Then Equation (2.7.12) follows. □

The following properties of $h_{n,K}$ and its derivatives will be proved in the Appendix in Section A.1.2 for $\gamma(n) = \Omega(\log(n))$.

(i) The function $\boldsymbol{s} \mapsto h_{n,K}(\boldsymbol{s})$ is infinitely often differentiable.

(ii) $\partial_{s_k} h_{n,K}(\mathbf{0}) = \frac{1}{\gamma(n)} \sum_{j=1}^{b_{t_k}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \sim t_k \frac{n}{\gamma(n)\alpha(n)}$.

(iii) $\partial_{s_{k_2}} \partial_{s_{k_1}} h_{n,K}(\mathbf{0}) \sim \frac{n/\alpha(n)}{(\gamma(n))^2} t_{k_2}(1-t_{k_1})$ for $k_2 \leq k_1$.

(iv) $\partial_{s_{k_2}} \partial_{s_{k_1}} h_{n,K}(\boldsymbol{s}) = \mathcal{O}\left(\frac{n}{(\gamma(n))^2\alpha(n)}\right)$ locally uniformly in $\boldsymbol{s}$.

(v) $\partial_{s_{k_3}} \partial_{s_{k_2}} \partial_{s_{k_1}} h_{n,K}(\boldsymbol{s}) = \mathcal{O}\left(\frac{n}{(\gamma(n))^3\alpha(n)}\right)$ locally uniformly in $\boldsymbol{s}$.

Hence, by Equation (2.7.9), we obtain

$$M_{n,K}(\boldsymbol{s}) = \exp\left(\nabla h_{n,K}(\mathbf{0}) \cdot \boldsymbol{s} + \mathcal{O}\left(\frac{n}{(\gamma(n))^2 \alpha(n)} |\boldsymbol{s}|^2\right)\right)(1+o(1)) \tag{2.7.17}$$

and

$$\begin{aligned} &M_{n,K}(\boldsymbol{s}) \\ &= \exp\left(\nabla h_{n,K}(\mathbf{0}) \cdot \boldsymbol{s} + \frac{1}{2}\langle \boldsymbol{s}, H_{h_{n,K}}(\mathbf{0})\boldsymbol{s}\rangle + \mathcal{O}\left(\frac{n}{(\gamma(n))^3 \alpha(n)} |\boldsymbol{s}|^3\right)\right)(1+o(1)), \end{aligned} \tag{2.7.18}$$

where $H_{h_{n,K}}$ denotes the Hessian matrix of $h_{n,K}$ and the error terms in the arguments of the exponential functions are locally uniform in $\boldsymbol{s}$.
Let first $\gamma(n) := n/\alpha(n) = \Omega(\log(n))$. Then, by Equation (2.7.17),

$$\lim_{n\to\infty} \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\exp\left(\sum_{k=1}^{M} \frac{s_k}{n/\alpha(n)} K_{b_{t_k}(n)}\right)\right] = \lim_{n\to\infty} M_{n,K}(\boldsymbol{s}) = \exp\left(\sum_{k=1}^{M} s_k t_k\right) \tag{2.7.19}$$

for $s_k \geq 0$.
If $\gamma(n) = \sqrt{n/\alpha(n)} = \Omega(\log(n))$, Equation (2.7.18) leads to

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\exp\left(\sum_{k=1}^{M} s_k L_{t_k}(n)\right)\right] \\ &= \lim_{n\to\infty} \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\exp\left(\sum_{k=1}^{M} \frac{s_k}{\sqrt{n/\alpha(n)}}\left(K_{b_{t_k}(n)} - \sum_{j=1}^{b_{t_k}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j\right)\right)\right] \\ &= \lim_{n\to\infty} M_{n,K}(\boldsymbol{s}) \exp\left(-\nabla h_{n,K}(\mathbf{0}) \cdot \boldsymbol{s}\right) \\ &= \exp\left(\frac{1}{2}\langle \boldsymbol{s}, A(\boldsymbol{t})\boldsymbol{s}\rangle\right), \end{aligned} \tag{2.7.20}$$

where $A(\boldsymbol{t}) = (A_{k_1,k_2})_{1\leq k_1,k_2\leq M}$ is a symmetric matrix with entries

$$A_{k_1,k_2} = t_{k_2}(1-t_{k_1})$$

for $k_2 \leq k_1$. Note that $A(\boldsymbol{t})$ is the covariance matrix of the Brownian bridge.
By Corollary 1.2.7, the pointwise convergence of the respective moment-generating functions entails that the joint distributions of $\left(\frac{K_{b_{t_k}(n)}}{n/\alpha(n)}\right)_{k=1}^{M}$ and $(L_{t_k}(n))_{k=1}^{M}$ converge weakly to a constant and the finite-dimensional distribution of the Brownian bridge, respectively.
We can now give the

Proof of Theorem 2.7.1. In the following, we use arguments of the proof of Corollary 3.4 in [**16**]. Let $\epsilon > 0$ and choose $0 = t_0 < t_1 < \cdots < t_M = 1$ such that $t_{k+1} - t_k < \frac{\epsilon}{2}$ for all k. Then, by monotonicity of $K_{b_t(n)}$ in t,

$$\left|\frac{K_{b_t(n)}}{n/\alpha(n)} - t\right| > \epsilon$$

for some $t \in [0,1]$ necessitates that there is some k such that

$$\left|\frac{K_{b_{t_k}(n)}}{n/\alpha(n)} - t_k\right| > \frac{\epsilon}{2}.$$

Consequently, we have

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\sup_{t\in[0,1]}\left|\frac{K_{b_t(n)}}{n/\alpha(n)}-t\right|>\epsilon\right]\leq\sum_{k=1}^{M}\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\left|\frac{K_{b_{t_k}(n)}}{n/\alpha(n)}-t_k\right|>\frac{\epsilon}{2}\right]. \tag{2.7.21}$$

By Equation (2.7.19),

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\left|\frac{K_{b_{t_k}(n)}}{n/\alpha(n)}-t_k\right|>\frac{\epsilon}{2}\right]$$

converges to 0 for all k. The claim follows. □

In order to establish convergence in distribution of

$$(L_t(n))_{t\in[0,1]}=\left(\frac{K_{b_t(n)}-\sum_{j=1}^{b_t(n)}\frac{\vartheta}{j}x_{n,\vartheta}^j}{\sqrt{n/\alpha(n)}}\right)_{t\in[0,1]}$$

to the Brownian bridge as a stochastic process, we have to prove weak convergence of the finite-dimensional distributions and tightness. Since weak convergence of the finite-dimensional distributions has already been shown, the remainder of the proof of Theorem 2.7.5 deals with the problem of tightness. For this purpose, we have to introduce a modification of $b_t(n)$ given by

$$c_t(n):=\begin{cases}\left\lfloor\alpha(n)+\log(t)\frac{\alpha(n)}{\log\left(\frac{n}{\alpha(n)}\right)}\right\rfloor & \text{if } \frac{1}{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2}\leq t\leq 1\\ \left\lfloor t\cdot\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2\alpha(n)\left(1-2\frac{\log\log\left(\frac{n}{\alpha(n)}\right)}{\log\left(\frac{n}{\alpha(n)}\right)}\right)\right\rfloor & \text{if } 0\leq t<\frac{1}{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2}\end{cases}$$

for $0\leq t\leq 1$ and $n\in\mathbb{N}$. The introduction of $c_t(n)$ is necessary since the following lemma does not hold if we substitute $b_t(n)$ for $c_t(n)$.

LEMMA 2.7.15. *There are $N\in\mathbb{N}$, $r\in\mathbb{N}$, and $C>0$ such that*

$$\sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)}\frac{\vartheta}{j}x_{n,\vartheta}^j\leq C\frac{n}{\alpha(n)}\left(t_2^{\frac{1}{r}}-t_1^{\frac{1}{r}}\right)$$

for all $n\geq N$ and $0\leq t_1<t_2\leq 1$ satisfying $c_{t_2}(n)-c_{t_1}(n)\geq 2$.

Due to its technical nature, the proof of Lemma 2.7.15 will be provided in the Appendix in Section A.1.3. We can now give the

PROOF OF THEOREM 2.7.5. We are going to apply Proposition 1.2.8 in order to show that

$$\left(\bar{L}_t(n)\right)_{t\in[0,1]}:=\left(\frac{K_{c_t(n)}-\sum_{j=1}^{c_t(n)}\frac{\vartheta}{j}x_{n,\vartheta}^j}{\sqrt{n/\alpha(n)}}\right)_{t\in[0,1]}$$

converges in distribution to the Brownian bridge. One can show that this entails the statement of Theorem 2.7.5 since

$$\sup_{\sigma\in S_{n,\alpha}}d_1\left(\left(\bar{L}_t(n)(\sigma)\right)_{t\in[0,1]},(L_t(n)(\sigma))_{t\in[0,1]}\right)\leq\frac{1}{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2}\xrightarrow{n\to\infty}0. \tag{2.7.22}$$

Equation (2.7.22) holds since there is $\lambda_n\in\Lambda$ (see Section 1.2.3) such that $b_t(n)=c_{\lambda_n(t)}(n)$ for all n and t which satisfies $\|\lambda_n\|_\infty\leq\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2}$ by the definition of $c_t(n)$. Note further that, if we fix $t>0$, we have $b_t(n)=c_t(n)$ for large n since $\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2}\to 0$. So the finite-dimensional distributions of $\left(\bar{L}_t(n)\right)_{t\in[0,1]}$ also converge to those of the Brownian bridge. It thus suffices to prove that there are $N\in\mathbb{N}$, $r\in\mathbb{N}$, and $C>0$ such that

$$I_n:=\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\left|\bar{L}_t(n)-\bar{L}_{t_1}(n)\right|^2\left|\bar{L}_{t_2}(n)-\bar{L}_t(n)\right|^2\right]\leq C\left(t_2^{\frac{1}{r}}-t_1^{\frac{1}{r}}\right)^2 \tag{2.7.23}$$

for all $n \geq N$ and $0 \leq t_1 \leq t \leq t_2 \leq 1$. By definition,

$$I_n = \mathbb{E}_{n,\alpha}^{(\vartheta)} \left[\left(\frac{K_{c_t(n)} - K_{c_{t_1}(n)} - \sum_{j=c_{t_1}(n)+1}^{c_t(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j}{\sqrt{n/\alpha(n)}} \right)^2 \cdot \left(\frac{K_{c_{t_2}(n)} - K_{c_t(n)} - \sum_{j=c_t(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j}{\sqrt{n/\alpha(n)}} \right)^2 \right].$$

By Equation (1.3.6), we have

$$\begin{aligned} F_n(s_1, s_2) :=& \mathbb{E}_{n,\alpha}^{(\vartheta)} \left[\exp\left(s_1 \frac{K_{c_t(n)} - K_{c_{t_1}(n)}}{\sqrt{n/\alpha(n)}} + s_2 \frac{K_{c_{t_2}(n)} - K_{c_t(n)}}{\sqrt{n/\alpha(n)}} \right) \right] \\ =& \frac{1}{Z_{n,\alpha,\vartheta}} [z^n] \exp\left(\sum_{j=1}^{c_{t_1}(n)} \frac{\vartheta}{j} z^j + \sum_{j=c_{t_1}(n)+1}^{c_t(n)} \frac{\vartheta e^{\sqrt{\frac{\alpha(n)}{n}} s_1}}{j} z^j \right. \\ & \left. + \sum_{j=c_t(n)+1}^{c_{t_2}(n)} \frac{\vartheta e^{\sqrt{\frac{\alpha(n)}{n}} s_2}}{j} z^j + \sum_{j=c_{t_2}(n)+1}^{\alpha(n)} \frac{\vartheta}{j} z^j \right). \end{aligned}$$

Since F_n is smooth, we obtain

$$\partial_{s_1}^{m_1} \partial_{s_2}^{m_2} F_n(s_1, s_2)\big|_{(s_1,s_2)=\mathbf{0}} = \mathbb{E}_{n,\alpha}^{(\vartheta)} \left[\left(\frac{K_{c_t(n)} - K_{c_{t_1}(n)}}{\sqrt{n/\alpha(n)}} \right)^{m_1} \left(\frac{K_{c_{t_2}(n)} - K_{c_t(n)}}{\sqrt{n/\alpha(n)}} \right)^{m_2} \right]$$

for $m_1, m_2 \in \mathbb{N}$. Hence, by applying linearity of the expectation and first expanding and then refactorizing the products, we can rewrite I_n as

$$\begin{aligned} I_n =& \left(\partial_{s_1} - \frac{\sum_{j=c_{t_1}(n)+1}^{c_t(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j}{\sqrt{n/\alpha(n)}} \right)^2 \left(\partial_{s_2} - \frac{\sum_{j=c_t(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j}{\sqrt{n/\alpha(n)}} \right)^2 F_n(s_1, s_2) \Bigg|_{(s_1,s_2)=\mathbf{0}} \\ =& \frac{1}{Z_{n,\alpha,\vartheta}} \left(\frac{\alpha(n)}{n} \right)^2 [z^n] G_{n,t_1,t}(z) G_{n,t,t_2}(z) \exp\left(\sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} z^j \right), \end{aligned} \tag{2.7.24}$$

where

$$G_{n,t_1,t}(z) := \left(\sum_{j=c_{t_1}(n)+1}^{c_t(n)} \frac{\vartheta}{j} \left(z^j - x_{n,\vartheta}^j \right) \right)^2 + \sum_{j=c_{t_1}(n)+1}^{c_t(n)} \frac{\vartheta}{j} z^j. \tag{2.7.25}$$

The extra term $\sum_{j=c_{t_1}(n)+1}^{c_t(n)} \frac{\vartheta}{j} z^j$ in Equation (2.7.25) results from the product rule when one calculates the second derivative with respect to s_1 (an analogous statement holds for s_2). The goal is to apply the saddle-point method to Equation (2.7.24). The array $\boldsymbol{q}$ with $q_{j,n} = \vartheta$ is admissible by Lemma 2.2.4. If we consider

$$f_n(z) := G_{n,t_1,t}(z) G_{n,t,t_2}(z),$$

then f_n satisfies conditions (1) and (2) in Definition 2.1.2. To see this, note that

$$f_n(x_{n,\vartheta}) = \left(\sum_{j=c_{t_1}(n)+1}^{c_t(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \right) \left(\sum_{j=c_t(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \right)$$

and

$$\begin{aligned} &|f_n(z)| \\ \leq& \left(\left(\sum_{j=c_{t_1}(n)+1}^{c_t(n)} \frac{2\vartheta}{j} x_{n,\vartheta}^j \right)^2 + \sum_{j=c_{t_1}(n)+1}^{c_t(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \right) \left(\left(\sum_{j=c_t(n)+1}^{c_{t_2}(n)} \frac{2\vartheta}{j} x_{n,\vartheta}^j \right)^2 + \sum_{j=c_t(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \right) \end{aligned}$$

for $|z| = x_{n,\vartheta}$, then apply Lemma 2.7.14. However, we cannot expect Condition (3) to hold in general since the highest power of z occurring in f_n is asymptotically equivalent to $(\alpha(n))^4$. So we cannot apply Proposition 2.1.4 directly. Instead, we are going to retrace the steps in the proof of Proposition 2.1.4 while performing a number of suitable modifications. So, by Cauchy's formula, we express I_n in Equation (2.7.24) as a contour integral along $\partial B_{x_{n,\vartheta}}(0)$ and define $g_n(\varphi) := \sum_{j=1}^{\alpha(n)} \vartheta x_{n,\vartheta}^j \frac{\mathrm{e}^{\mathrm{i}j\varphi}-1}{j}$. We thus obtain

$$I_n = \frac{(\alpha(n))^2}{Z_{n,\alpha,\vartheta} n^2} \frac{\exp\left(\sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j\right)}{2\pi x_{n,\vartheta}^n} \int_{-\pi}^{\pi} G_{n,t_1,t}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right) G_{n,t,t_2}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right) \exp\left(g_n(\varphi)\right) \mathrm{d}\varphi.$$

In the following, C will denote a positive generic constant which may change its value from line to line. Since $G_{n,t_1,t}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right) G_{n,t,t_2}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right) = 0$ whenever $c_{t_2}(n) - c_{t_1}(n) < 2$, we only have to consider t_1 and t_2 such that

$$c_{t_2}(n) - c_{t_1}(n) \geq 2. \tag{2.7.26}$$

We again split the integral into two parts. Since f_n satisfies condition (2), tails pruning still works and the contribution from $[-\pi,\pi] \setminus [-\varphi_n, \varphi_n]$ with $\varphi_n = n^{-\frac{5}{12}}(\alpha(n))^{-\frac{7}{12}}$ decays faster than any power of $\frac{1}{n}$ (cf. the proof of Proposition 2.1.4). By Equation (2.7.26) and the definition of f_n, the relevant contribution will therefore be from the interval $[-\varphi_n, \varphi_n]$. If we use

$$\left|\mathrm{e}^{\mathrm{i}j\varphi} - 1\right| \leq Cj\varphi$$

and

$$\left|\mathrm{e}^{\mathrm{i}j\varphi}\right| = 1,$$

which hold for all $1 \leq j \leq \alpha(n)$ and $\varphi \in [-\varphi_n, \varphi_n]$,

$$\begin{aligned}
&\left|G_{n,t_1,t}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right) G_{n,t,t_2}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right)\right| \\
\leq& C\left(\left(\varphi \sum_{j=b_{t_1}(n)+1}^{b_t(n)} \vartheta x_{n,\vartheta}^j\right)^2 + \sum_{j=b_{t_1}(n)+1}^{b_t(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j\right) \\
&\cdot \left(\left(\varphi \sum_{j=b_t(n)+1}^{b_{t_2}(n)} \vartheta x_{n,\vartheta}^j\right)^2 + \sum_{j=b_t(n)+1}^{b_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j\right) \\
\leq& C\left(\left(\varphi \sum_{j=b_{t_1}(n)+1}^{b_{t_2}(n)} \vartheta x_{n,\vartheta}^j\right)^2 + \sum_{j=b_{t_1}(n)+1}^{b_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j\right)^2 \\
\leq& C\left(\left(\varphi\alpha(n) \sum_{j=b_{t_1}(n)+1}^{b_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j\right)^2 + \sum_{j=b_{t_1}(n)+1}^{b_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j\right)^2
\end{aligned} \tag{2.7.27}$$

follows. We further have

$$\left|\exp\left(g_n(\varphi)\right)\right| \leq C\exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right)$$

for all $\varphi \in [-\varphi_n, \varphi_n]$ by Equation (2.1.7). So, due to the finite moments of the normal distribution, substituting v for $\sqrt{\lambda_{2,n}}\varphi$ yields

$$\int_{-\varphi_n}^{\varphi_n} \varphi^k \exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right) \mathrm{d}\varphi \leq C\frac{1}{\lambda_{2,n}^{\frac{k+1}{2}}}.$$

Recall that $\lambda_{2,n} \sim n\alpha(n)$ by Lemma 2.3.1. Altogether, by expanding the products in Equation (2.7.27) and applying Lemmata 2.3.1 and 2.7.15, we have

$$\begin{aligned}
|I_n| \leq & C\frac{\sqrt{\lambda_{2,n}}\,(\alpha(n))^2}{n^2}\int_{-\pi}^{\pi}\left|G_{n,t_1,t_2}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right)G_{n,t_1,t_2}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right)\exp\left(g_n(\varphi)\right)\right|\mathrm{d}\varphi \\
\leq & C\frac{(\alpha(n))^2}{n^2}\left(\left(t_2^{\frac{1}{r}}-t_1^{\frac{1}{r}}\right)^2\frac{n^2}{\lambda_{2,n}}+\left(t_2^{\frac{1}{r}}-t_1^{\frac{1}{r}}\right)\frac{n}{\alpha(n)}\right)^2 \\
\leq & C\left(\left(t_2^{\frac{1}{r}}-t_1^{\frac{1}{r}}\right)^2+\left(t_2^{\frac{1}{r}}-t_1^{\frac{1}{r}}\right)\right)^2 \\
\leq & C\left(t_2^{\frac{1}{r}}-t_1^{\frac{1}{r}}\right)^2
\end{aligned}$$

for $n \geq N$. Here, the last line is a consequence of $0 \leq t_1 < t_2 \leq 1$. This completes the proof. □

REMARK 2.7.16. In the linear parametrization given by $\tilde{b}_\tau(n)$ it is not necessary to use an analogue of $c_t(n)$ and one may consider $(\tau_2 - \tau_1)$ instead of $\left(\tau_1^{\frac{1}{r}} - \tau_2^{\frac{1}{r}}\right)$ for some large $r \in \mathbb{N}$ in Equation (2.7.23) since one can limit the considerations to $\tau_1, \tau_2 \in [0, T]$ for some $T > 0$ (cf. Proposition 1.2.9): In this case one can then apply $\tilde{b}_\tau(n) \sim \alpha(n)$ uniformly in $\tau \in [0, T]$ and one further avoids the situation present in the logarithmic parametrization that $(\log(t))' = \frac{1}{t}$ becomes large for small t.

2.7.3. Proofs of Theorems 2.7.2 and 2.7.6. We start by giving the proof of Theorem 2.7.2, which can be deduced from Theorems 2.7.1 and 2.7.5. It could also be proved in the same way as Theorem 2.7.1.

PROOF OF THEOREM 2.7.2. By retracing the beginning of the proof of Theorem 2.7.1, one can deduce the following analogue of Equation (2.7.21): For $\epsilon > 0$ there exist $0 \leq t_1 < t_2 < \cdots < t_M \leq 1$ such that

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\sup_{t\in[0,1]}\left|\frac{S_{b_t(n)}}{n}-t\right|>\epsilon\right]\leq\sum_{k=1}^{M}\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\left|\frac{S_{b_{t_k}(n)}}{n}-t_k\right|>\frac{\epsilon}{2}\right]$$

holds. Since $S_{b_0(n)} = 0$ and $S_{b_1(n)} = n$ by definition, we can restrict the discussion to the case of $0 < t_k < 1$.

Let $K_{b_t(n),\alpha(n)} := \sum_{j=b_t(n)+1}^{\alpha(n)} C_j$ and $S_{b_t(n),\alpha(n)} := \sum_{j=b_t(n)+1}^{\alpha(n)} jC_j$ for $t \in (0, 1)$. Then,

$$\begin{aligned}
\left|\frac{S_{b_t(n)}}{n}-t\right| &= \left|1-t-\frac{S_{b_t(n),\alpha(n)}}{n}\right| \\
&= \left|1-t-\frac{K_{b_t(n),\alpha(n)}}{n/\alpha(n)}\delta(n)\right|,
\end{aligned}$$

where $\delta(n)$ is a random variable which satisfies

$$\frac{b_t(n)}{\alpha(n)} \leq \delta(n) \leq 1.$$

By the triangle inequality, we have

$$\begin{aligned}
\left|\frac{S_{b_t(n)}}{n}-t\right| &\leq \left|\frac{K_{b_t(n)}}{n/\alpha(n)}-t\right|+\left|1-\frac{K_{\alpha(n)}}{n/\alpha(n)}\right|+\left|\frac{K_{b_t(n),\alpha(n)}}{n/\alpha(n)}(\delta(n)-1)\right| \\
&\leq \left|\frac{K_{b_t(n)}}{n/\alpha(n)}-t\right|+\left|1-\frac{K_{\alpha(n)}}{n/\alpha(n)}\right|+\left|\frac{K_{\alpha(n)}}{n/\alpha(n)}(\delta(n)-1)\right|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\left|\frac{S_{b_t(n)}}{n}-t\right|>\frac{\epsilon}{2}\right] \leq & \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\left|\frac{K_{b_t(n)}}{n/\alpha(n)}-t\right|>\frac{\epsilon}{6}\right]+\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\left|1-\frac{K_{\alpha(n)}}{n/\alpha(n)}\right|>\frac{\epsilon}{6}\right] \\
&+\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\left|\frac{K_{\alpha(n)}}{n/\alpha(n)}(\delta(n)-1)\right|>\frac{\epsilon}{6}\right].
\end{aligned} \tag{2.7.28}$$

Since $b_t(n) \sim \alpha(n)$ by definition, $\delta(n) \to 1$ uniformly (as a function defined on $S_{n,\alpha}$). So, by Theorems 2.7.1 and 2.7.5, the individual terms in Equation (2.7.28) converge to 0. The claim is proved. $\square$

In order to prove Theorem 2.7.6, we follow the approach in Section 2.7.2 and consider the finite-dimensional distributions of $\left(S_{b_t(n)}\right)_{t\in[0,1]}$. Let again $\gamma = (\gamma(n))_{n\in\mathbb{N}}$ be a sequence which satisfies $\gamma(n) \to \infty$ as $n \to \infty$, $\boldsymbol{s} = (s_k)_{k=1}^M$ and $\boldsymbol{t} = (t_k)_{k=1}^M$ such that $0 \le t_1 < t_2 < \cdots < t_M \le 1$. Also define $t_0 := 0$ and $t_{M+1} := 1$. Then, by Equation (1.3.6), we ontain for the relevant moment-generating function that

$$\begin{aligned} M_{n,S}(\boldsymbol{s}) :=& \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\prod_{k=1}^M \exp\left(\frac{s_k}{\gamma(n)} S_{b_{t_k}(n)}\right)\right] \\ =& \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\prod_{k=1}^M \exp\left(\frac{s_k}{\gamma(n)} \sum_{j=1}^{b_{t_k}(n)} jC_j\right)\right] \\ =& \frac{1}{Z_{n,\alpha,\vartheta}}\left[z^n\right] \exp\left(\sum_{k=0}^M \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \frac{\vartheta}{j}\left(\mathrm{e}^{\sum_{i=k+1}^M \frac{s_i}{\gamma(n)}} z\right)^j\right). \end{aligned} \tag{2.7.29}$$

The goal is to apply Proposition 2.1.4 to Equation (2.7.29). We set $f_n(z) = 1$, so the perturbations are admissible. Let the triangular array $\boldsymbol{q}$ be given by

$$q_{j,n} = \vartheta \mathrm{e}^{\sum_{i=k(j,n)}^M \frac{js_i}{\gamma(n)}},$$

where $k(j,n) := \min\{1 \le i \le K : b_{t_i}(n) \ge j\}$. Intuitively, this means that any index i with $b_{t_i}(n) \ge j$ contributes a factor of $\mathrm{e}^{js_i/\gamma(n)}$ to $q_{j,n}$ since $S_{b_{t_i}(n)}$ includes the indices in cycles of length j in this case. Hence, the new saddle point $x_{n,S}(\boldsymbol{s}) := x_{n,\boldsymbol{q}}$ is the unique positive solution of

$$n = \sum_{k=0}^M \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta\left(\mathrm{e}^{\sum_{i=k+1}^M \frac{s_i}{\gamma(n)}} x_{n,S}(\boldsymbol{s})\right)^j. \tag{2.7.30}$$

Note that $x_{n,S}(\boldsymbol{0}) = x_{n,\vartheta}$. We have to verify admissibility for the array $\boldsymbol{q}$, which will be done in the following Lemmata 2.7.17 and 2.7.18.

LEMMA 2.7.17. *Let γ be a sequence, $\boldsymbol{t} = (t_k)_{1\le k\le M}$ with $0 = t_0 < t_1 < ... < t_i < ... < t_{M+1} = 1$, and $s_k \ge 0$ for all $1 \le k \le M$. Then we have*

$$\alpha(n) \log\left(\mathrm{e}^{\sum_{k=1}^M \frac{s_k}{\gamma(n)}} x_{n,S}(\boldsymbol{s})\right) \sim \log\left(\frac{n}{\alpha(n)}\right) \tag{2.7.31}$$

uniformly in $\boldsymbol{s}$. In particular,

$$\lim_{n\to\infty} \mathrm{e}^{\sum_{k=1}^M \frac{s_k}{\gamma(n)}} x_{n,S}(\boldsymbol{s}) = 1 \tag{2.7.32}$$

uniformly in $\boldsymbol{s}$.

PROOF. By Equation (2.7.30), we deduce from $s_k \ge 0$ for all k that

$$x_{n,\alpha}\left(\frac{n}{\vartheta\alpha(n)}\right) \le \mathrm{e}^{\sum_{k=1}^M \frac{s_k}{\gamma(n)}} x_{n,S}(\boldsymbol{s}) \le x_{n,b_{t_1}(n)}\left(\frac{n}{\vartheta b_{t_1}(n)}\right). \tag{2.7.33}$$

By applying Lemma 2.2.1 to both $x_{n,\alpha}\left(\frac{n}{\vartheta\alpha(n)}\right)$ and $x_{n,b_{t_1}(n)}\left(\frac{n}{\vartheta\alpha(n)}\right)$, Equation (2.7.31) follows after a short calculation due to $b_{t_1}(n) \sim \alpha(n)$. Equation (2.7.32) is a direct consequence of Equation (2.7.33) and the fact that $\lim x_{n,\alpha}\left(\frac{n}{\vartheta\alpha(n)}\right) = 1 = \lim x_{n,b_{t_1}(n)}\left(\frac{n}{\vartheta b_{t_1}(n)}\right)$ according to Lemma 2.2.1. $\square$

LEMMA 2.7.18. *Let $\gamma(n)=\Omega(\alpha(n))$ and $\boldsymbol{t}=(t_1,...,t_M)^T$ with $0=t_0\le t_1<...<t_M\le t_{M+1}=1$ for $M\in\mathbb{N}$. Then, locally uniformly in $\boldsymbol{s}=(s_1,...,s_M)^T\in[0,\infty)^M$,*

$$\lambda_{2,n}=\sum_{k=0}^{M}\sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)}\vartheta j\left(\mathrm{e}^{\sum_{i=k+1}^{M}\frac{s_i}{\gamma(n)}}x_{n,S}(\boldsymbol{s})\right)^j=n\alpha(n)+O\left(\frac{n\alpha(n)}{\log(n)}\right).$$

PROOF. W.l.o.g., let $0<t_1$ and $t_M<1$. In this proof, we use the shorthands $x:=x_{n,S}(\boldsymbol{s})$ and $E_k:=\mathrm{e}^{\sum_{i=k+1}^{M}\frac{s_i}{\gamma(n)}}$ for the sake of simplicity. We then have

$$\begin{aligned}\lambda_{2,n}=&\sum_{k=0}^{M}\sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)}\vartheta j\left(E_kx\right)^j\\=&\vartheta x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{M}\sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)}\left(E_kx\right)^j\\=&\vartheta x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{M}\left(E_kx\right)^{b_{t_k}(n)+1}\frac{1-\left(E_kx\right)^{b_{t_{k+1}}(n)-b_{t_k}(n)}}{1-E_kx},\end{aligned}$$

where the last line applies the formula for the geometric series. By calculating the derivative, we arrive at three terms due to Leibniz's rule. The first one is given by

$$\begin{aligned}T_1=&\vartheta\sum_{k=0}^{M}\left[b_{t_k}(n)+1\right]\left(E_kx\right)^{b_{t_k}(n)+1}\frac{1-\left(E_kx\right)^{b_{t_{k+1}}(n)-b_{t_k}(n)}}{1-E_kx}\\=&\sum_{k=0}^{M}\left[b_{t_k}(n)+1\right]\sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)}\vartheta\left(E_kx\right)^j.\end{aligned}$$

The remaining terms are given by

$$\begin{aligned}T_2=&-\vartheta\sum_{k=0}^{M}\left(b_{t_{k+1}}(n)-b_{t_k}(n)\right)\frac{\left(E_kx\right)^{b_{t_{k+1}}(n)+1}}{1-E_kx}\\=&-\vartheta b_{t_1}(n)\frac{\left(E_0x\right)^{b_{t_1}(n)+1}}{1-E_0x}\\&-\vartheta\sum_{k=1}^{M}\left(b_{t_{k+1}}(n)-b_{t_k}(n)\right)\frac{\left(E_kx\right)^{b_{t_{k+1}}(n)+1}}{1-E_kx}\end{aligned}$$

and

$$\begin{aligned}T_3=&\vartheta\sum_{k=0}^{M}\left(E_kx\right)^{b_{t_k}(n)+2}\frac{1-\left(E_kx\right)^{b_{t_{k+1}}(n)-b_{t_k}(n)}}{\left(1-E_kx\right)^2}\\=&\sum_{k=0}^{M}\frac{E_kx}{1-E_kx}\sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)}\vartheta\left(E_kx\right)^j,\end{aligned}$$

respectively. Since $\gamma(n)$ diverges as $n\to\infty$, we have $E_kx\to1$ locally uniformly in $\boldsymbol{s}$. So,

$$\begin{aligned}\frac{1}{1-E_kx}\sim-&\frac{1}{\log\left(E_kx\right)}=-\frac{1}{\log\left(\mathrm{e}^{\sum_{i=k+1}^{M}\frac{s_i}{\gamma(n)}}x\right)}\\=&-\frac{1}{\log\left(E_0x\right)-\sum_{i=1}^{k}s_i/\gamma(n)}\\=&\mathcal{O}\left(\frac{\alpha(n)}{\log(n)}\right)\end{aligned}\tag{2.7.34}$$

for all k locally uniformly in $\boldsymbol{s}$ by Lemma 2.7.17 and since $\gamma(n) = \Omega(\alpha(n))$. By Equation (2.7.30), we conclude that

$$T_3 = \sum_{k=0}^{M} \mathcal{O}\left(\frac{\alpha(n)}{\log(n)}\right) \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta (E_k x)^j = \mathcal{O}\left(\frac{n\alpha(n)}{\log(n)}\right).$$

By Equation (2.7.3), we have

$$b_{t_{k+1}}(n) - b_{t_k}(n) = \mathcal{O}\left(\frac{\alpha(n)}{\log(n)}\right)$$

for all $k > 0$. Hence,

$$\begin{aligned} T_2 = & -\vartheta b_{t_1}(n) \frac{(E_0 x)^{b_{t_1}(n)+1}}{1 - E_0 x} + \mathcal{O}\left(\frac{n\alpha(n)}{\log(n)}\right) \\ = & b_{t_1}(n) \sum_{j=1}^{b_{t_1}(n)} \vartheta (E_0 x)^j + \mathcal{O}\left(\frac{n\alpha(n)}{\log(n)}\right) \end{aligned}$$

follows by also applying Equations (2.7.34) and (2.7.30) as well as the formula for the geometric series. Moreover, a similar argument yields

$$T_1 = \sum_{k=1}^{M} [b_{t_k}(n) + 1] \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta (E_k x)^j + \mathcal{O}\left(\frac{n\alpha(n)}{\log(n)}\right).$$

Altogether, we have

$$\begin{aligned} \lambda_{2,n} = & T_1 + T_2 + T_3 \\ = & \sum_{k=1}^{M} [b_{t_k}(n) + 1] \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta (E_k x)^j + b_{t_1}(n) \sum_{j=1}^{b_{t_1}(n)} \vartheta (E_0 x)^j + \mathcal{O}\left(\frac{n\alpha(n)}{\log(n)}\right). \end{aligned}$$

Due to Equation (2.7.3), $b_{t_k}(n) = \alpha(n) + \mathcal{O}(\alpha(n) / \log(n))$ for all $k \geq 1$. Thus,

$$\begin{aligned} \lambda_{2,n} = & \sum_{k=1}^{M} b_{t_k}(n) \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta (E_k x)^j + b_{t_1}(n) \sum_{j=1}^{b_{t_1}(n)} \vartheta (E_0 x)^j + \mathcal{O}\left(\frac{n\alpha(n)}{\log(n)}\right) \\ = & n\alpha(n) + \mathcal{O}\left(\frac{n\alpha(n)}{\log(n)}\right) \end{aligned}$$

by Equation (2.7.30), and the claim is proved. □

Lemmata 2.7.12 and 2.7.13 show that $\boldsymbol{q}$ fulfils conditions (1) and (2) in Definition 2.1.1 if $\gamma(n) = \Omega(\alpha(n))$. The third condition follows immediately from the definition of the array, so $\boldsymbol{q}$ is admissible. By applying Proposition 2.1.4 and Lemma 2.7.18 to Equation (2.7.29), we conclude that

$$M_{n,S}(\boldsymbol{s}) = \frac{1}{Z_{n,\alpha,\vartheta}} \frac{1}{\sqrt{2\pi n\alpha(n)}} \exp(h_{n,S}(\boldsymbol{s}))(1 + o(1))$$

holds pointwise in $\boldsymbol{s} \in [0, \infty)^M$, where

$$h_{n,S}(\boldsymbol{s}) := \sum_{k=0}^{M} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \frac{\vartheta}{j} \left(\exp\left(\sum_{i=k+1}^{M} \frac{s_i}{\gamma(n)}\right) x_{n,S}(\boldsymbol{s})\right)^j - n \log(x_{n,S}(\boldsymbol{s})).$$

Note that we can again infer

$$Z_{n,\alpha,\vartheta} \sim \frac{1}{\sqrt{2\pi n\alpha(n)}} \exp(h_{n,S}(\boldsymbol{0})) \tag{2.7.35}$$

from $M_{n,S}(\boldsymbol{0}) = 1$. In order to expand $h_{n,S}$ about $\boldsymbol{s} = \boldsymbol{0}$, we need the asymptotics of the derivatives. The ensuing properties of $h_{n,S}$ and its derivatives will be proved in Section A.1.2 for $\gamma(n) = \Omega(\alpha(n))$.

(i) The function $\boldsymbol{s} \mapsto h_{n,S}(\boldsymbol{s})$ is infinitely often differentiable.

(ii) $\partial_{s_k} h_{n,S}(\mathbf{0}) = \frac{1}{\gamma(n)} \sum_{j=1}^{b_{t_k}(n)} \vartheta x_{n,\vartheta}^j \sim t_k \frac{n}{\gamma(n)}$.

(iii) $\partial_{s_{k_2}} \partial_{s_{k_1}} h_{n,S}(\mathbf{0}) \sim \frac{n\alpha(n)}{(\gamma(n))^2} t_{k_2}(1 - t_{k_1})$ for $k_2 \le k_1$.

(iv) $\partial_{s_{k_3}} \partial_{s_{k_2}} \partial_{s_{k_1}} h_{n,S}(\boldsymbol{s}) = \mathcal{O}\left(\frac{n(\alpha(n))^2}{(\gamma(n))^3}\right)$ locally uniformly in $\boldsymbol{s}$.

Thus, Equation (2.7.35) entails

$$M_{n,S}(\boldsymbol{s}) = \exp\left(\nabla h_{n,S}(\mathbf{0}) \cdot \boldsymbol{s} + \frac{1}{2}\langle \boldsymbol{s}, H_{h_{n,S}}(\mathbf{0})\boldsymbol{s}\rangle + \mathcal{O}\left(\frac{n(\alpha(n))^2}{(\gamma(n))^3}|\boldsymbol{s}|^3\right)\right)(1 + o(1)). \tag{2.7.36}$$

Note that the error term in the argument of the exponential function is locally uniform in $\boldsymbol{s}$.

REMARK 2.7.19. We could now prove Theorem 2.7.2 in the same way as we did Theorem 2.7.1, but we omit it here because another proof has already been given.

Let $\gamma(n) = \sqrt{n\alpha(n)} = \Omega(\alpha(n))$. Then, by Equation (2.7.36), we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\exp\left(\sum_{k=1}^{M} s_k \tilde{L}_{t_k}(n)\right)\right] \\
&= \lim_{n\to\infty} \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\exp\left(\sum_{k=1}^{M} \frac{s_k}{\sqrt{n\alpha(n)}}\left(S_{b_{t_k}(n)} - \sum_{j=1}^{b_{t_k}(n)} \vartheta x_{n,\vartheta}^j\right)\right)\right] \\
&= \lim_{n\to\infty} M_{n,S}(\boldsymbol{s}) \exp\left(-\nabla h_{n,K}(\mathbf{0}) \cdot \boldsymbol{s}\right) \\
&= \exp\left(\frac{1}{s}\langle \boldsymbol{s}, A(\boldsymbol{t})\boldsymbol{s}\rangle\right), \qquad (2.7.37)
\end{aligned}$$

Recall that $A(\boldsymbol{t}) = (A_{k_1,k_2})_{1\le k_1,k_2\le M}$ is a symmetric matrix with entries

$$A_{k_1,k_2} = t_{k_2}(1 - t_{k_1})$$

for $k_2 \le k_1$. It is the covariance matrix of the Brownian bridge. By Corollary 1.2.7, we therefore conclude that the finite-dimensional distributions of $\left(\tilde{L}_t(n)\right)_{t\in[0,1]}$ converge weakly to those of the Brownian bridge. As in Section 2.7.2, in order to conclude the proof of Theorem 2.7.6, we have to establish tightness. In this case, one could dispense with introducing the functions $c_t(n)$, but if we do use $c_t(n)$ we do not have to show additional technical lemmata.

PROOF OF THEOREM 2.7.6. We basically have to repeat the proof of Theorem 2.7.5, so we only highlight the differences that have to be accounted for. Since we apply the same criteria, we have to show this time that

$$\left(\hat{L}_t(n)\right)_{t\in[0,1]} := \left(\frac{S_{c_t(n)} - \sum_{j=1}^{c_t(n)} \vartheta x_{n,\vartheta}^j}{\sqrt{n\alpha(n)}}\right)_{t\in[0,1]}$$

converges in distribution to the Brownian bridge. The goal is to prove that there are $C > 0$, $r \in \mathbb{N}$, and $N \in \mathbb{N}$ such that

$$I_n := \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\left|\hat{L}_t(n) - \hat{L}_{t_1}(n)\right|^2 \left|\hat{L}_{t_2}(n) - \hat{L}_t(n)\right|^2\right] \le C\left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right)^2$$

holds for all $n \ge N$ and all $0 \le t_1 \le t \le t_2$. In the following, C denotes a generic constant whose value may change throughout the proof. With

$$\begin{aligned}
F_n(s_1, s_2) :=& \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\exp\left(s_1 \frac{S_{c_t(n)} - S_{c_{t_1}(n)}}{\sqrt{n\alpha(n)}} + s_2 \frac{S_{c_{t_2}(n)} - S_{c_t(n)}}{\sqrt{n\alpha(n)}}\right)\right] \\
=& \frac{1}{Z_{n,\alpha,\vartheta}}[z^n]\exp\left(\sum_{j=1}^{c_{t_1}(n)} \frac{\vartheta}{j} z^j + \sum_{j=c_{t_1}(n)+1}^{c_t(n)} \frac{\vartheta}{j}\left(\mathrm{e}^{\frac{s_1}{\sqrt{n\alpha(n)}}} z\right)^j\right. \\
&\left. + \sum_{j=c_t(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j}\left(\mathrm{e}^{\frac{s_2}{\sqrt{n\alpha(n)}}} z\right)^j + \sum_{j=c_{t_2}(n)+1}^{\alpha(n)} \frac{\vartheta}{j} z^j\right)
\end{aligned}$$

by Equation (1.3.6), we obtain

$$
\begin{aligned}
I_n =& \left(\partial_{s_1} - \frac{\sum_{j=c_{t_1}(n)+1}^{c_t(n)} \vartheta x_{n,\vartheta}^j}{\sqrt{n\alpha(n)}}\right)^2 \left(\partial_{s_2} - \frac{\sum_{j=c_t(n)+1}^{c_{t_2}(n)} \vartheta x_{n,\vartheta}^j}{\sqrt{n\alpha(n)}}\right)^2 F_n(s_1,s_2)\Bigg|_{(s_1,s_2)=\mathbf{0}} \\
=& \frac{(n\alpha(n))^{-2}}{Z_{n,\alpha,\vartheta}} [z^n] G_{n,t_1,t}(z) G_{n,t,t_2}(z) \exp\left(\sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} z^j\right),
\end{aligned}
$$

where

$$
G_{n,t_1,t}(z) := \left(\sum_{j=c_{t_1}(n)+1}^{c_t(n)} \vartheta\left(z^j - x_{n,\vartheta}^j\right)\right)^2 + \sum_{j=c_{t_1}(n)+1}^{c_t(n)} \vartheta j z^j.
$$

Hence,

$$
I_n = \frac{(n\alpha(n))^{-2}}{Z_{n,\alpha,\vartheta}} \frac{\exp\left(\sum_{j=1}^{\alpha(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j\right)}{2\pi x_{n,\vartheta}^n} \int_{-\pi}^{\pi} G_{n,t_1,t}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right) G_{n,t,t_2}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right) \exp\left(g_n(\varphi)\right) \mathrm{d}\varphi.
$$

Due to tails pruning, we only have to consider the contribution from $[-\varphi_n, \varphi_n]$. Since

$$
\left|G_{n,t_1,t}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right) G_{n,t,t_2}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right)\right| \le C \left(\left(\varphi \sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \vartheta j x_{n,\vartheta}^j\right)^2 + \sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \vartheta j x_{n,\vartheta}^j\right)^2
$$

and

$$
\sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \vartheta j x_{n,\vartheta}^j \le (\alpha(n))^2 \sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j
$$

hold in this case, we arrive at

$$
\begin{aligned}
I_n \le& C \frac{\sqrt{\lambda_{2,n}}}{(n\alpha(n))^2} \int_{-\pi}^{\pi} \left|G_{n,t_1,t_2}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right) G_{n,t_1,t_2}\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right) \exp\left(g_n(\varphi)\right)\right| \mathrm{d}\varphi \\
\le& C (n\alpha(n))^{-2} \left(\left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right)^2 \frac{(n\alpha(n))^2}{\lambda_{2,n}} + \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right) n\alpha(n)\right)^2 \\
\le& C \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right)^2.
\end{aligned}
$$

The claim is proved. □

2.8. Longest Cycles

In this section, we investigate the behaviour of the longest cycles for the different regimes discussed in Section 2.3.2. Aside from one technical condition which we believe not to be necessary (see Proposition 2.8.2 and Remark 2.8.3), we will see that the limit distributions of the longest cycles only depend on the asymptotic behaviour of the expected number of cycles of maximal length, $\mu_{\alpha(n)}(n) = \mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{\alpha(n)}\right]$. The three relevant regimes are characterized by $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{\alpha(n)}\right] \to \infty$, $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{\alpha(n)}\right] \to \mu > 0$, and $\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[C_{\alpha(n)}\right] \to 0$, respectively. Before proceeding to the different cases, we give a weaker result which states that the lengths of the longest cycles are asymptotically equivalent to $\alpha(n)$ in all three regimes.

Propositions 2.8.1 and 2.8.2 haven been given in [**10**] for $\vartheta = 1$. Sections 2.8.3 and 2.8.4 offer new results.

For the whole section we assume that Equation (2.0.2) holds.

2.8.1. General Case. Let $l_k = l_k(\sigma)$ denote the length of the k-th longest cycle of the permutation σ. The general result, which holds for all three regimes, is a consequence of the existence of the limit shape for cumulative cycle numbers proved in Theorem 2.7.1.

Proposition 2.8.1 ([**10**, Theorem 2.7]). *Let α be as in Equation (2.0.2) and $K \in \mathbb{N}$. Then, under the probability measures $\mathbb{P}_{n,\alpha}^{(\vartheta)}$, we have*

$$\frac{1}{\alpha(n)}\left(l_1, l_2, \ldots, l_K\right) \xrightarrow{d} (1, 1, \ldots, 1)$$

as $n \to \infty$.

Proof. By definition of $K_{b_t(n)}$ (see Section 2.7),

$$\begin{aligned}\left|\frac{1}{n/\alpha(n)}\left(\sum_{j=b_t(n)+1}^{\alpha(n)} C_j\right) - (1-t)\right| &= \left|\frac{K_{\alpha(n)} - K_{b_t(n)}}{n/\alpha(n)} - (1-t)\right| \\ &\leq \left|\frac{K_{\alpha(n)}}{n/\alpha(n)} - 1\right| + \left|\frac{K_{b_t(n)}}{n/\alpha(n)} - t\right|\end{aligned}$$

holds for all $t \in [0,1]$. If we apply Theorem 2.7.1 with $t = \frac{1}{2}$, we thus obtain for $\epsilon > 0$ that

$$\begin{aligned}&\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\left|\frac{1}{n/\alpha(n)}\left(\sum_{j=b_{1/2}(n)+1}^{\alpha(n)} C_j\right) - \frac{1}{2}\right| > \epsilon\right] \\ \leq &\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\left|\frac{K_{\alpha(n)}}{n/\alpha(n)} - 1\right| > \frac{\epsilon}{2}\right] + \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\left|\frac{K_{b_{1/2}(n)}}{n/\alpha(n)} - \frac{1}{2}\right| > \frac{\epsilon}{2}\right] \\ \to &0\end{aligned}$$

as $n \to \infty$. In particular, the probability that there are more than $\left(\frac{1}{2} - \epsilon\right)\frac{n}{\alpha(n)}$ cycles of lengths larger than $b_{1/2}(n)$ tends to 1 as $n \to \infty$. Note that $\frac{n}{\alpha(n)}$ diverges. So, since $b_{1/2}(n) \sim \alpha(n)$, the claim follows. □

2.8.2. Case of Divergence of $\mu_{\alpha(n)}(n)$.

Proposition 2.8.2 ([**10**, Theorem 2.8]). *Let $\alpha(n) = \mathcal{O}\left(n^{\frac{1}{2}}\right)$ and $\alpha(n) \geq n^{\frac{1}{7}+\delta}$ for some $\delta > 0$. Then, for each $K \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[(l_1, l_2, \ldots, l_K) \neq (\alpha(n), \alpha(n), \ldots, \alpha(n))\right] = 0.$$

Proof. We apply Corollary 2.4.10. Since $\mu_{\alpha(n)}(n)$ diverges as n tends to infinity (cf. Section 2.3.2), we conclude that

$$\frac{C_{\alpha(n)} - \mu_{\alpha(n)}(n)}{\sqrt{\mu_{\alpha(n)}(n)}}$$

converges in distribution to the standard normal distribution. Hence,

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[C_{\alpha(n)} < K\right] = \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\frac{C_{\alpha(n)} - \mu_{\alpha(n)}(n)}{\sqrt{\mu_{\alpha(n)}(n)}} < \frac{K - \mu_{\alpha(n)}(n)}{\sqrt{\mu_{\alpha(n)}(n)}}\right] \xrightarrow{n\to\infty} 0,$$

and the claim follows. $\square$

REMARK 2.8.3. Note that we only assume $\alpha(n) \geq n^{\frac{1}{7}+\delta}$ in order to apply Corollary 2.4.10. Since we do not believe this condition to be necessary (see the discussion in Remark 2.4.11), we expect Proposition 2.8.2 to hold for all $\alpha = \mathcal{O}\left(n^{\frac{1}{2}}\right)$ which satisfy Equation (2.0.2).

2.8.3. Case of Convergence of $\mu_{\alpha(n)}(n)$ to a Positive Number.

PROPOSITION 2.8.4. *Let α be as in Equation (2.0.2). If*

$$\mu_{\alpha(n)}(n) \xrightarrow{n\to\infty} \mu > 0, \tag{2.8.1}$$

we have

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[l_k = \alpha(n) - d\right] \xrightarrow{n\to\infty} \frac{1}{\Gamma(k)} \int_{d\mu}^{(d+1)\mu} v^{k-1} \mathrm{e}^{-v} \mathrm{d}v$$

for $d \in \mathbb{N}_0$. In other words, $\alpha(n) - l_k$ converges in distribution to $\lfloor \mu^{-1} X \rfloor$, where X is a gamma-distributed random variable with parameters k and 1.

PROOF. Let $i \in \mathbb{N}_0$. Then Equation (2.3.4) implies that

$$\frac{\mu_{\alpha(n)}(n)}{\mu_{\alpha(n)-i}(n)} = \frac{\alpha(n) - i}{\alpha(n)} x_{n,\vartheta}^{i} \xrightarrow{n\to\infty} 1.$$

Hence, we deduce

$$\mu_{\alpha(n)-i}(n) \xrightarrow{n\to\infty} \mu$$

for all $i \in \mathbb{N}_0$ from Equation (2.8.1). Consider independent Poisson-distributed random variables $(Z_i)_{0\leq i\leq I}$ with parameters $\mathbb{E}[Z_i] = \mu$ for all i. By Proposition 2.4.1, the cycle counts $\left(C_{\alpha(n)-i}\right)_{0\leq i\leq I}$ under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ converge in distribution to $(Z_i)_{0\leq i\leq I}$. In particular,

$$\lim_{n\to\infty} \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\left(C_{\alpha(n)-i}\right)_{0\leq i\leq I} = \boldsymbol{c}\right] = \prod_{i=0}^{I} \mathbb{P}\left[Z_i = c_i\right]$$

holds for all $\boldsymbol{c} = (c_i)_{i=0}^{I} \in \mathbb{N}_0^{I+1}$. Note that one can conclude that

$$\begin{aligned} \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[l_k \leq \alpha(n) - d\right] =& \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\sum_{i=0}^{d-1} C_{\alpha(n)-i} \leq k-1\right] \\ \to& \mathbb{P}\left[\sum_{i=0}^{d-1} Z_i \leq k-1\right] \end{aligned}$$

as $n \to \infty$. By independence of $(Z_i)_{0\leq i\leq I}$, the random variable $\sum_{i=0}^{d-1} Z_i$ is Poisson-distributed with parameter $d\mu$. Thus,

$$\mathbb{P}\left[\sum_{i=0}^{d-1} Z_i \leq k-1\right] = \sum_{j=0}^{k-1} \mathrm{e}^{-d\mu} \frac{(d\mu)^j}{j!} = \frac{1}{\Gamma(k)} \int_{d\mu}^{\infty} v^{k-1} \mathrm{e}^{-v} \mathrm{d}v$$

by [**1**, Equation (26.4.21)]. Here, Γ denotes the gamma function. Since

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[l_k = \alpha(n) - d\right] = \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[l_k \leq \alpha(n) - d\right] - \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[l_k \leq \alpha(n) - (d+1)\right],$$

we have

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[l_k = \alpha(n) - d\right] \xrightarrow{n\to\infty} \frac{1}{\Gamma(k)} \int_{d\mu}^{(d+1)\mu} v^{k-1} \mathrm{e}^{-v} \mathrm{d}v.$$

The claim is proved. $\square$

REMARK 2.8.5. In principle, the proof of Proposition 2.8.4 enables us to also compute the limit of

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[(l_k)_{k=1}^{K} = (\alpha(n) - d_k)_{k=1}^{K}\right]$$

as n tends to infinity since the event in question only depends on a finite number of cycle counts $\left(C_{\alpha(n)-i}\right)_{0\leq i\leq I}$ which converge to independent Poisson-distributed random variables with parameters μ. It is, however, cumbersome to provide a closed form for such probabilities: The reason for this is that the stochastic process (l_k) is not Markovian, i.e. the distribution of l_K depends non-trivially on the distribution of the random vector $(l_k)_{k=1}^{K-1}$. This is why we only provide the readily interpretable results for one individual l_k at a time in the proposition.

2.8.4. Case of Convergence of $\mu_{\alpha(n)}(n)$ to 0. The first result in this section establishes convergence of the cumulative numbers of long cycles on a certain scale to a Poisson process. This scale is closer to $\alpha(n)$ than $b_t(n)$ defined in Section 2.7 since it identifies at what point cycles first appear, whereas $b_t(n)$ is concerned with fractions of the total number of cycles. By considering the jump times of the Poisson process, which in this picture correspond to the lengths of longest cycles, it is then possible to prove limit theorems for l_k.
Let

$$d_t(n) := \max\left\{\alpha(n) - \left\lfloor \frac{t}{\mu_{\alpha(n)}(n)} \right\rfloor, 0\right\}.$$

THEOREM 2.8.6. *Let α be as in Equation (2.0.2). We have*

$$\mu_{\alpha(n)}(n) \xrightarrow{n\to\infty} 0$$

if and only if

$$\lim_{n\to\infty} \frac{n\log(n)}{(\alpha(n))^2} = 0. \tag{2.8.2}$$

In this case, the law of

$$\left(\hat{K}_t(n)\right)_{t\in[0,\infty)} := \left(\sum_{j=d_t(n)+1}^{\alpha(n)} C_j\right)_{t\in[0,\infty)}$$

under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ converges weakly in $\mathcal{D}[0,\infty)$ to a Poisson process with parameter 1. Here,

$$\mu_{\alpha(n)}(n) \sim \frac{n}{\vartheta(\alpha(n))^2}\log\left(\frac{n}{\alpha(n)}\right)$$

as n tends to infinity.

PROOF. Let $\mu_n := \mu_{\alpha(n)}(n)$ for the sake of brevity. By Lemma 2.3.1 and Equation (2.0.2), we have

$$\mu_n \sim \frac{n}{\vartheta(\alpha(n))^2}\log\left(\frac{n}{\alpha(n)}\right) \approx \frac{n\log(n)}{(\alpha(n))^2}. \tag{2.8.3}$$

So $\mu_n \to 0$ if and only if Equation (2.8.2) holds, which proves the first statement.
Assume in the following that α satisfies Equation (2.8.2). We will first prove certain auxiliary results. Let $t \geq 0$ and n large enough, then

$$\mu_{d_t(n)}(n) = \frac{x_{n,\vartheta}^{\alpha(n)-\lfloor t/\mu_n\rfloor}}{\alpha(n) - \lfloor t/\mu_n\rfloor} = \mu_n \frac{\alpha(n)}{\alpha(n) - \lfloor t/\mu_n\rfloor} x_{n,\vartheta}^{-\lfloor t/\mu_n\rfloor}$$

holds. Note that

$$\frac{\alpha(n)}{\alpha(n) - \lfloor t/\mu_n\rfloor} \xrightarrow{n\to\infty} 1$$

locally uniformly in t since $1/\mu_n = o(\alpha(n))$ by Equation (2.8.3). By Lemma 2.3.1 and Equation (2.8.3), we have

$$\begin{aligned} x_{n,\vartheta}^{-\lfloor t/\mu_n\rfloor} \sim x_{n,\vartheta}^{-t/\mu_n} &\sim \exp\left(-\frac{t}{\alpha(n)\mu_n}\log\left(\frac{n}{\alpha(n)\vartheta}\log\left(\frac{n}{\alpha(n)\vartheta}\right)\right)\right) \\ &= \exp\left(\mathcal{O}\left(t\frac{\alpha(n)}{n}\right)\right) \\ &\to 1 \end{aligned}$$

locally uniformly in $t \geq 0$. Altogether, we conclude

$$\mu_{\alpha(n)-\lfloor t/\mu_n\rfloor}(n) \sim \mu_n$$

locally uniformly in t. Since $\mu_j(n)$ is non-decreasing in j for $\alpha(n) - \lfloor t/\mu_n \rfloor \leq j \leq \alpha(n)$ if n is large enough (cf. Section 2.3.2), we obtain

$$\sum_{j=\alpha(n)-\lfloor t/\mu_n\rfloor+1}^{\alpha(n)} \mu_j(n) \xrightarrow{n\to\infty} \frac{t}{\mu_n}\mu_n = t \tag{2.8.4}$$

locally uniformly in t.

In order to establish convergence as stochastic processes, we begin by proving convergence of the finite-dimensional distributions. More precisely, for $0 = t_0 \leq t_1 < \ldots < t_K$ and $K \in \mathbb{N}$, consider the increments

$$\left(\hat{K}_{t_k}(n) - \hat{K}_{t_{k-1}}(n)\right)_{k=1}^{K}.$$

By Equation (1.3.6), for n large enough, the relevant moment-generating function is given by

$$\begin{aligned} &\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\prod_{k=1}^{K} \exp\left(s_k\left(\hat{K}_{t_k}(n) - \hat{K}_{t_{k-1}}(n)\right)\right)\right] \\ =&\frac{1}{Z_{n,\alpha,\vartheta}}\left[z^n\right] \exp\left(\sum_{k=1}^{K}\left(\mathrm{e}^{s_k}-1\right)\sum_{j=\alpha(n)-\lfloor t_k/\mu_n\rfloor+1}^{\alpha(n)-\lfloor t_{k-1}/\mu_n\rfloor} \frac{\vartheta}{j}z^j\right) \exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}z^j\right). \end{aligned} \tag{2.8.5}$$

Let $s_k \geq 0$ for all $1 \leq k \leq K$. We want to apply Proposition 2.1.4 with the array $\boldsymbol{q}_\vartheta$ and the perturbations

$$f_n(z) = \exp\left(\sum_{k=1}^{K}\left(\mathrm{e}^{s_k}-1\right)\sum_{j=\alpha(n)-\lfloor t_k/\mu_n\rfloor+1}^{\alpha(n)-\lfloor t_{k-1}/\mu_n\rfloor} \frac{\vartheta}{j}z^j\right).$$

Recall that $\boldsymbol{q}_\vartheta$ is admissible by Lemma 2.2.4. Since f_n is entire for all z and

$$|f_n(z)| \leq f_n(x_{n,\vartheta})$$

for all $|z| = x_{n,\vartheta}$ due to $s_k \geq 0$, it only remains to check condition (3) in Definition 2.1.2. We have

$$f_n'(z) = \sum_{k=1}^{K}\left(\mathrm{e}^{s_k}-1\right)\sum_{j=\alpha(n)-\lfloor t_k/\mu_n\rfloor+1}^{\alpha(n)-\lfloor t_{k-1}/\mu_n\rfloor} \vartheta z^{j-1} f_n(z),$$

so

$$\frac{|f_n'(z)|}{f_n(x_{n,\vartheta})} \leq \sum_{k=1}^{K}\left(\mathrm{e}^{s_k}-1\right)\sum_{j=\alpha(n)-\lfloor t_k/\mu_n\rfloor+1}^{\alpha(n)-\lfloor t_{k-1}/\mu_n\rfloor} \vartheta x_{n,\vartheta}^{j-1}$$

for all $|z| = x_{n,\vartheta}$. Admissibility of f_n is then a consequence of

$$\begin{aligned} \sum_{k=1}^{K}\left(\mathrm{e}^{s_k}-1\right)\sum_{j=\alpha(n)-\lfloor t_k/\mu_n\rfloor+1}^{\alpha(n)-\lfloor t_{k-1}/\mu_n\rfloor} \vartheta x_{n,\vartheta}^{j-1} \leq& \sum_{k=1}^{K}\left(\mathrm{e}^{s_k}-1\right)\sum_{j=\alpha(n)-\lfloor t_k/\mu_n\rfloor+1}^{\alpha(n)-\lfloor t_{k-1}/\mu_n\rfloor} \vartheta x_{n,\vartheta}^{\alpha(n)} \\ =&\mathcal{O}\left(\frac{x_{n,\vartheta}^{\alpha(n)}}{\mu_n}\right) \\ \subset&\mathcal{O}(\alpha(n)) \\ \subset&o\left(n^{-\frac{5}{12}}(\alpha(n))^{-\frac{7}{12}}\right). \end{aligned}$$

Observe that Equation (2.8.4) entails

$$\sum_{j=\alpha(n)-\lfloor t_k/\mu_n\rfloor+1}^{\alpha(n)-\lfloor t_{k-1}/\mu_n\rfloor} \mu_j(n) \xrightarrow{n\to\infty} t_k - t_{k-1}$$

for all k. Thus, by Proposition 2.1.4 and Lemma 2.3.1, we obtain

$$\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\prod_{k=1}^{K}\exp\left(s_k\left(\hat{K}_{t_k}(n)-\hat{K}_{t_{k-1}}(n)\right)\right)\right]\sim f_n(x_{n,\vartheta})$$
$$\to\sum_{k=1}^{K}\exp\left[\left(\mathrm{e}^{s_k}-1\right)(t_k-t_{k-1})\right].$$

By Corollary 1.2.7, this implies that the increments

$$\left(\hat{K}_{t_k}(n)-\hat{K}_{t_{k-1}}(n)\right)_{k=1}^{K}$$

converge in distribution to independent random variables $(Z_1, Z_2, \dots, Z_K)$, where Z_k is Poisson-distributed with parameter $t_k - t_{k-1}$. In consequence, the finite-dimensional distributions of $\left(\hat{K}_t(n)\right)_{t\in[0,\infty)}$ converge weakly to the finite-dimensional distributions of the Poisson process with parameter 1.

By Proposition 1.2.9, it is sufficient to establish convergence of $\left(\hat{K}_t(n)\right)_{t\in[0,T]}$ to the Poisson process in $\mathcal{D}[0,T]$ for all $T>0$ in order to prove convergence of $\left(\hat{K}_t(n)\right)_{t\in[0,\infty)}$ to the Poisson process in $\mathcal{D}[0,\infty)$. Since we have already proved convergence of the finite-dimensional distributions in the general case, we assume w.l.o.g. $T=1$ and apply Proposition 1.2.8. It suffices to show

$$\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\left(\hat{K}_t(n)-\hat{K}_{t_1}(n)\right)^2\left(\hat{K}_{t_2}(n)-\hat{K}_t(n)\right)^2\right]=\mathcal{O}\left((t_2-t_1)^2\right)$$

uniformly in $0\le t_1\le t\le t_2\le 1$ since the limit is a Poisson process. Let n be large enough such that $d_1(n)>0$. By Equation (2.8.5), we have

$$\begin{aligned}
&\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\left(\hat{K}_t(n)-\hat{K}_{t_1}(n)\right)^2\left(\hat{K}_{t_2}(n)-\hat{K}_t(n)\right)^2\right]\\
=&\frac{\partial^2}{\partial s_2^2}\frac{\partial^2}{\partial s_1^2}\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\mathrm{e}^{s_1\left(\hat{K}_t(n)-\hat{K}_{t_1}(n)\right)+s_2\left(\hat{K}_{t_2}(n)-\hat{K}_t(n)\right)}\right]\Bigg|_{s=0}\\
=&\frac{1}{Z_{n,\alpha,\vartheta}}\frac{\partial^2}{\partial s_2^2}\frac{\partial^2}{\partial s_1^2}[z^n]\exp\left(\left(\mathrm{e}^{s_1}-1\right)G_{n,t_1,t}(z)+\left(\mathrm{e}^{s_2}-1\right)G_{n,t,t_2}(z)\right)\exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}z^j\right)\Bigg|_{s=0}\\
=&\frac{1}{Z_{n,\alpha,\vartheta}}[z^n]f_n(z)\exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}z^j\right),
\end{aligned}$$

where

$$G_{n,t,t_2}(z):=\sum_{j=\alpha(n)-\lfloor t_2/\mu_n\rfloor+1}^{\alpha(n)-\lfloor t/\mu_n\rfloor}\frac{\vartheta}{j}z^j$$

and

$$f_n(z):=G_{n,t_1,t}(z)\left(1+G_{n,t_1,t}(z)\right)G_{n,t,t_2}(z)\left(1+G_{n,t,t_2}(z)\right).$$

Note that $|f_n(z)|\le f_n(|z|)$ for all z.

We follow an approach similar to the one in Proposition 2.1.4 and apply intermediate results from its proof. Let again

$$g_n(\varphi)=\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}x_{n,\vartheta}^j\frac{\mathrm{e}^{\mathrm{i}j\varphi}-1}{j}.$$

By Cauchy's integral formula, we conclude that

$$\begin{aligned}
&\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\left(\hat{K}_t(n)-\hat{K}_{t_1}(n)\right)^2\left(\hat{K}_{t_2}(n)-\hat{K}_t(n)\right)^2\right]\\
=&\frac{1}{Z_{n,\alpha,\vartheta}}\frac{1}{2\pi\mathrm{i}}\int_{\partial B_{x_{n,\vartheta}}(0)} f_n(z)\exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}z^j\right)\frac{\mathrm{d}z}{z^{n+1}}\\
=&\frac{1}{Z_{n,\alpha,\vartheta}}\frac{\exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}x_{n,\vartheta}^j\right)}{2\pi x_{n,\vartheta}^n}\int_{-\pi}^{\pi} f_n\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right)\exp\left(g_n(\varphi)\right)\mathrm{d}\varphi.
\end{aligned}$$

The integral will be dominated by its central part, so we focus on the interval $[-\varphi_n,\varphi_n]$ with $\varphi_n=n^{-\frac{5}{12}}(\alpha(n))^{-\frac{7}{12}}$ first. Recall that, by Equation (2.1.7), there is $C>0$ such that

$$|\exp(g_n(\varphi))|\leq C\exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right) \tag{2.8.6}$$

for all $\varphi\in[\varphi_n,\varphi_n]$ and n large enough. Hence,

$$\begin{aligned}
\left|\int_{-\varphi_n}^{\varphi_n} f_n\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right)\exp(g_n(\varphi))\mathrm{d}\varphi\right| \leq& Cf_n(x_{n,\vartheta})\int_{-\varphi_n}^{\varphi_n}\exp\left(-\frac{\lambda_{2,n}}{2}\varphi^2\right)\\
\leq& Cf_n(x_{n,\vartheta})\frac{1}{\sqrt{\lambda_{2,n}}}.
\end{aligned}$$

Concerning the tails, we have

$$\left|\int_{\varphi_n\leq|\varphi|\leq\pi} f_n\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right)\exp(g_n(\varphi))\mathrm{d}\varphi\right|\leq f_n(x_{n,\vartheta})\int_{\varphi_n\leq|\varphi|\leq\pi}|\exp(g_n(\varphi))|\mathrm{d}\varphi,$$

of which we have already shown in the proof of Proposition 2.1.4 that it decreases faster than any power of n.
So let $N\in\mathbb{N}$ and $C'>0$ such that

$$\left|\int_{-\pi}^{\pi} f_n\left(x_{n,\vartheta}\mathrm{e}^{\mathrm{i}\varphi}\right)\exp(g_n(\varphi))\mathrm{d}\varphi\right|\leq C'f_n(x_{n,\vartheta})\sqrt{\frac{2\pi}{\lambda_{2,n}}},$$

$j\mapsto\mu_j(n)$ is increasing in $j\geq d_1(n)$, and

$$\frac{1}{Z_{n,\alpha,\vartheta}}\frac{\exp\left(\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}x_{n,\vartheta}^j\right)}{x_{n,\vartheta}^n\sqrt{2\pi\lambda_{2,n}}}\leq 2$$

for all $n\geq N$, where the last condition is possible due to Lemma 2.3.1. Now fix $n\geq N$. Since $\left(\hat{K}_t(n)-\hat{K}_{t_1}(n)\right)^2\left(\hat{K}_{t_2}(n)-\hat{K}_t(n)\right)^2=0$ unless $d_{t_2}(n)<d_t(n)<d_{t_1}(n)$, we only have to consider $t_1<t_2$ such that $t_2-t_1\geq\mu_n$ and $f_n(x_{n,\vartheta})>0$. We arrive at

$$\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\left(\hat{K}_t(n)-\hat{K}_{t_1}(n)\right)^2\left(\hat{K}_{t_2}(n)-\hat{K}_t(n)\right)^2\right]\leq C''f_n(x_{n,\vartheta})$$

for some $C''>0$ which does not depend on $n\geq N$. By definition of f_n, we have

$$\begin{aligned}
f_n(x_{n,\vartheta})\leq&\left(\sum_{j=d_{t_2(n)}+1}^{d_{t_1}(n)}\frac{\vartheta}{j}x_{n,\vartheta}^j\right)^2\left(1+\sum_{j=d_{t_2(n)}+1}^{d_{t_1}(n)}\frac{\vartheta}{j}x_{n,\vartheta}^j\right)^2\\
\leq&\left(d_{t_1}(n)-d_{t_2}(n)\right)^2\mu_n^2\left(1+\left(d_{t_1}(n)-d_{t_2}(n)\right)\mu_n\right)^2.
\end{aligned}$$

So there is $C''' > 0$ such that

$$\begin{aligned}
f_n\left(x_{n,\vartheta}\right) \leq & C'''\left(d_{t_1}(n) - d_{t_2}(n)\right)^2 \mu_n^2 \\
= & C'''\left(\left\lfloor \frac{t_2}{\mu_n} \right\rfloor - \left\lfloor \frac{t_1}{\mu_n} \right\rfloor\right)^2 \mu_n^2 \\
\leq & C'''\left(\frac{t_2}{\mu_n} - \frac{t_1}{\mu_n} + 1\right)^2 \mu_n^2 \\
\leq & 4C'''\left(t_2 - t_1\right)^2 .
\end{aligned}$$

Altogether, it follows that

$$\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\left(\hat{K}_t(n) - \hat{K}_{t_1}(n)\right)^2 \left(\hat{K}_{t_2}(n) - \hat{K}_t(n)\right)^2\right] \leq C''''\left(t_2 - t_1\right)^2$$

for some $C'''' > 0$ and all $n \geq N$. The claim is proved. $\square$

Corollary 2.8.7. *Let α satisfy Equation (2.0.2) and let $K \in \mathbb{N}$. Then, if*

$$\mu_{\alpha(n)}(n) \xrightarrow{n\to\infty} 0,$$

we have convergence in distribution of

$$\mu_{\alpha(n)}(n) \cdot \left(\alpha(n) - l_1, l_2 - l_1, \ldots, l_K - l_{K-1}\right)$$

under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ to independent exponentially distributed random variables with parameters 1. In particular, $\mu_{\alpha(n)}(n)\left(\alpha(n) - l_k\right)$ converges in distribution to a gamma-distributed random variable with parameters k and 1.

Proof. The claim is a consequence of the convergence established in the proof of Theorem 2.8.6 since the limit distribution is the distribution of the jump times of the Poisson process (see, e.g. [**44**, p. 5]). $\square$

2.9. Length of a Typical Cycle

When considering the problem of the distribution of the length of a typical cycle, two concepts have to be distinguished. On the one hand, it is possible to look at all cycles in a realization of the permutation and sample one of them independently and uniformly. We refer to the length of said cycle as J. On the other hand, one may fix an index, e.g. 1, and investigate the cycle $\tilde{J}_1$ which contains 1. Due to different lengths of the cycles, in general these approaches lead to different results. In the case of random permutations without macroscopic cycles, however, the limit distributions in both cases coincide. This is a consequence of the limit shapes of both cumulative cycle and index numbers, which move exactly in parallel.

Proposition 2.9.1. *Let α be as in Equation (2.0.2). We have*

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[J\le b_t(n)\right]\\ =&\lim_{n\to\infty}\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\tilde{J}_1\le b_t(n)\right]\\ =&t\end{aligned}$$

for $t\in[0,1]$. In particular, $\frac{\alpha(n)-J}{\alpha(n)/\log(n/\alpha(n))}$ and $\frac{\alpha(n)-\tilde{J}_1}{\alpha(n)/\log(n/\alpha(n))}$ converge in distribution to an exponentially distributed random variable with parameter 1 as n tends to infinity.

Proof. Note that, by symmetry, $\tilde{J}_1$ and $\tilde{J}_j$ have the same distribution under $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ for all $1\le j\le n$. Hence,

$$\begin{aligned}\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\tilde{J}_1\le b_t(n)\right]=&\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\mathbb{1}_{\left\{\tilde{J}_1\le b_t(n)\right\}}\right]\\ =&\frac{1}{n}\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\sum_{j=1}^{n}\mathbb{1}_{\left\{\tilde{J}_j\le b_t(n)\right\}}\right]\\ =&\frac{1}{n}\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[S_{b_t(n)}\right],\end{aligned}$$

where $S_{b_t(n)}$ has been defined in Section 2.7. Since $0\le\frac{S_{b_t(n)}}{n}\le 1$, we conclude from Theorem 2.7.2 that

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\tilde{J}_1\le b_t(n)\right]\xrightarrow{n\to\infty}t.$$

A similar approach works in the case of J. Recall that l_k denotes the length of the kth longest cycle. With $K_{b_t(n)}$ as in Section 2.7, we arrive at

$$\begin{aligned}\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[J\le b_t(n)\right]=&\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\mathbb{1}_{\{J\le b_t(n)\}}\right]\\ =&\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\frac{1}{K_{\alpha(n)}}\sum_{k=1}^{K_{\alpha(n)}}\mathbb{1}_{\{l_k\le b_t(n)\}}\right]\\ =&\mathbb{E}_{n,\alpha}^{(\vartheta)}\left[\frac{K_{b_t(n)}}{K_{\alpha(n)}}\right].\end{aligned}$$

Due to $0\le\frac{K_{b_t(n)}}{K_{\alpha(n)}}\le 1$,

$$\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[J\le b_t(n)\right]\xrightarrow{n\to\infty}t$$

is then a consequence of Theorem 2.7.1. Concerning the second statement, let $s\ge 0$ and consider

$$\begin{aligned}\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\frac{\alpha(n)-J}{\alpha(n)/\log\left(\frac{n}{\alpha(n)}\right)}\le s\right]=&\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[J\ge\alpha(n)-s\frac{\alpha(n)}{\log\left(\frac{n}{\alpha(n)}\right)}\right]\\ =&\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[J\ge b_{\mathrm{e}^{-s}}(n)\right]\\ =&1-\mathbb{P}_{n,\alpha}^{(\vartheta)}\left[J<b_{\mathrm{e}^{-s}}(n)\right]\end{aligned}$$

for large n. Hence, we have

$$\lim_{n\to\infty} \mathbb{P}_{n,\alpha}^{(\vartheta)}\left[\frac{\alpha(n) - J}{\alpha(n) / \log\left(\frac{n}{\alpha(n)}\right)} \leq s\right] = 1 - \mathrm{e}^{-s}.$$

Since the same argument applies to $\tilde{J}_1$, the claim is proved. $\square$

CHAPTER 3

Outlook

3.1. Spatial and Surrogate-Spatial Random Permutations

In the following sections, we are going to review parts of the three papers [**11, 15, 23**] dealing with two models which can be considered as instances of random permutations with cycle weights depending on the system size. These models are spatial and surrogate-spatial random permutations and have already been referred to in the introduction.
Due to the change of context, different conventions may apply to certain variables than in other sections of this thesis.

3.1.1. Spatial Random Permutations. The model of spatial random permutations arises in the context of quantum statistical mechanics and is related to the topic of Bose-Einstein condensation. Nevertheless, it is a classical probabilistic model and thus belongs to the set of approaches which try to address problems in quantum mechanics by extracting and investigating pertinent probability measures (see, e.g., [**62**] for a further example of such a strategy).
The state space of the model is given by

$$\Omega_{\Lambda,n} = \Lambda^n \times S_n,$$

where $\Lambda = [0, L]^d \subset \mathbb{R}^d$ for some $L > 0$, $n \in \mathbb{N}$, and $d \in \mathbb{N}$. The domain Λ^n may be thought of as the configuration space of the positions of n particles and thus constitutes the (d-dimensional) spatial component of the model.
Let $V : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\int_{\mathbb{R}^d} \exp(-V(x))\,\mathrm{d}x = 1$ and define $V_\Lambda(x)$ by

$$\mathrm{e}^{-V_\Lambda(x)} = \sum_{w \in \mathbb{Z}^d} \mathrm{e}^{-V(x+Lw)},$$

which will serve as a tool to implement boundary conditions sometimes referred to as periodized.
The probability measure on $\Omega_{\Lambda,n}$ is then the Gibbs state with respect to the Hamiltonian

$$H(\boldsymbol{x}, \sigma) = \sum_{i=1}^{n} V_\Lambda\left(x_i - x_{\sigma(i)}\right) + \sum_{j=1}^{n} \delta_j C_j(\sigma)$$

for $\boldsymbol{x} = (x_i)_{i=1}^n \in \Lambda^n$ and $\sigma \in S_n$, where $(\delta_j)_j$ is a sequence of weights. Explicitly, the Gibbs state is then given by

$$\mathbb{P}_\Lambda[\mathrm{d}\boldsymbol{x}, \{\sigma\}] = \frac{1}{n!Y}\mathrm{e}^{-H(\boldsymbol{x},\sigma)},$$

where Y is a normalizing constant and $\mathrm{d}\boldsymbol{x}$ denotes the Lebesgue measure on Λ^n.
If $\beta > 0$ is the inverse temperature and $V(x) = \frac{1}{4\beta}|x|^2 + \frac{1}{2}d\log(4\pi\beta)$, the model of spatial random permutations corresponds to the dilute Bose gas, which can be shown via its Feynman-Kac representation. The kinetic part of the Hamiltonian of the Bose gas is then encoded in V and the cycle weights $(\delta_j)_j$ are intended to approximate the influence of interaction between particles. Recall that bosons have symmetric wave functions, so the relevant partition function of the quantum model will be given by a certain trace in a symmetric subspace of the Fock space. Permutations then become salient as a device for computing said trace by explicitly symmetrizing the basis of a larger Hilbert space. Since Bose-Einstein condensation and the occurrence of macroscopic cycles have been related in certain instances (cf., e.g., [**47, 29, 60**]), one is justified in concentrating on the marginal

$$\mathbb{P}_{\Lambda,n}[\{\sigma\}] = \frac{1}{n!Y}\int_{\Lambda^n} \mathrm{e}^{-H(\boldsymbol{x},\sigma)}\mathrm{d}\boldsymbol{x} \tag{3.1.1}$$

of $\mathbb{P}_\Lambda$, the so-called annealed model. Clearly, we have

$$Y = \frac{1}{n!} \sum_{\sigma \in S_n} \int_{\Lambda^n} \mathrm{e}^{-H(\boldsymbol{x},\sigma)} \mathrm{d}\boldsymbol{x}. \tag{3.1.2}$$

Betz and Ueltschi in [**11**] are mainly interested in the question whether cycles of macroscopic lengths occur and how they are distributed. They therefore consider the thermodynamic limit (a concept from statistical mechanics) $L, n \to \infty$ where the density $\rho := n/|\Lambda|$ is fixed. Here, $|\Lambda|$ denotes the Lebesgue measure of Λ. Given $K > 0$, the expected fraction of points in cycles larger than K is

$$\nu_K := \liminf_{L,n \to \infty} \mathbb{E}_{\Lambda,n} \left[\frac{1}{n} \sum_{i : l_i > K} l_i \right],$$

where l_i denotes the ith longest cycle in a permutation. The fraction of points in infinite cycles is then given by the limit

$$\nu := \lim_{K \to \infty} \nu_K,$$

which can be shown to exist.
In order to establish their results, Betz and Ueltschi need to make certain technical assumptions about V, the most important one being that the Fourier transform

$$\mathrm{e}^{-\epsilon(k)} := \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} k \cdot x} \mathrm{e}^{-V(x)} \mathrm{d}x$$

is non-negative. In particular, this implies $V(x) = V(-x)$, $\epsilon(0) = 0$ by normalization, and $\epsilon(k) > 0$ for $k \neq 0$.
By defining the critical density as

$$\rho_c := \sum_{j=1} \mathrm{e}^{-\delta_j} \int_{\mathbb{R}^d} \mathrm{e}^{-j\epsilon(k)} \mathrm{d}k$$

and assuming certain properties of $(\delta_j)_j$, Betz and Ueltschi arrive at

PROPOSITION 3.1.1 ([**11**, Theorems 2.1 and 2.2]). *The fraction of points in infinite cycles is given by*

$$\nu = \max \left\{ 0, 1 - \frac{\rho_c}{\rho} \right\},$$

where ρ denotes the density parameter in the thermodynamic limit.

(1) *If $\rho > \rho_c$, i.e. $\nu > 0$, and $\delta_j = \delta \in \mathbb{R}$, the longest cycles converge in distribution to the Poisson-Dirichlet distribution: As $L, n \to \infty$, we have*

$$\left(\frac{l_1}{\nu n}, \frac{l_2}{\nu n}, \dots \right) \xrightarrow{d} \mathrm{PD}\left(\mathrm{e}^{-\delta}\right).$$

(2) *If $\rho > \rho_c$, i.e. $\nu > 0$, and $\delta_j = \gamma \log(j)$ for some $\gamma > 0$, then a single long cycle occurs: As $L, n \to \infty$, we have*

$$\frac{l_1}{\nu n} \xrightarrow{d} 1.$$

Depending on V and the spatial dimension d, the critical density ρ_c may be finite or infinite. In the Gaussian case, ρ_c is finite for $d \geq 3$. Concerning the cycle weights, we restrict the discussion to the case of $\delta_j = \delta \in \mathbb{R}$. Note that the assumption in Proposition 3.1.1(1) may actually be relaxed a little. Recall the convergence of the lengths of the longest cycles under the Ewens measure to the Poisson-Dirichlet distribution discussed in Section 1.1.3. Intuitively, the result in Proposition 3.1.1 suggests that one can divide the permuted indices into two groups: One group of indices does not lie in long cycles, its fraction of all indices is $\min\left\{1, \frac{\rho_c}{\rho}\right\}$. The other group of indices, however, behaves as though belonging to an $\mathrm{e}^{-\delta}$-biased random permutation, its fraction being $1 - \min\left\{1, \frac{\rho_c}{\rho}\right\} = \nu$. We give a sketch of the proof aimed at supporting this intuition and focus on the connection of the model to random permutations with cycle weights.

Let $\sigma \in S_n$ and consider the definition of $\mathbb{P}_{\Lambda,n}\left[\{\sigma\}\right]$. One sees that the integrals factorize according to the cycles, and the contribution of a cycle of length j is

$$\mathrm{e}^{-\delta} \int_{\Lambda^j} \mathrm{e}^{-\sum_{i=1}^{j} V_\Lambda(y_i - y_{i+1})} \prod_{i=1}^{j} \mathrm{d}y_i = \mathrm{e}^{-\delta} \left|\Lambda\right| \sum_{w \in \mathbb{Z}^d} \left(\mathrm{e}^{-V}\right)^{*j} (Lw)\,, \tag{3.1.3}$$

where $y_{j+1} := y_1$ and $\left(\mathrm{e}^{-V}\right)^{*j}$ is the j-fold convolution of $\left(\mathrm{e}^{-V}\right)$ with itself. Equation (3.1.3) can be shown by starting from the right-hand side, applying $|\Lambda| = \int_\Lambda \mathrm{d}y_1$, and shifting all the variables in the integrals of the convolutions by y_1. If one further applies the Poisson summation formula (see, e.g., [**32**, Theorem 3.2.8])

$$\sum_{w \in \mathbb{Z}^d} f(Lw) = \frac{1}{|\Lambda|} \sum_{k \in \mathbb{Z}^d/L} \hat{f}(k)\,,$$

where $\hat{f}$ denotes the Fourier transform of f, one obtains

$$\mathbb{P}_{\Lambda,n}\left[\{\sigma\}\right] = \frac{1}{n!Y} \prod_{j=1}^{n} \left(\mathrm{e}^{-\delta} \sum_{k \in \mathbb{Z}^d/L} \mathrm{e}^{-j\epsilon(k)} \right)^{C_j(\sigma)} \tag{3.1.4}$$

since

$$\widehat{\left(\mathrm{e}^{-V}\right)^{*j}} = \mathrm{e}^{-j\epsilon(k)}.$$

We have thus identified the cycle weights

$$q_{j,n} = \mathrm{e}^{-\delta} \sum_{k \in \mathbb{Z}^d/L} \mathrm{e}^{-j\epsilon(k)}$$

which correspond to $\mathbb{P}_{\Lambda,n}$. By Cauchy's formula in Equation (1.1.3), we calculate

$$\mathbb{P}_{\Lambda,n}\left[\boldsymbol{C} = \boldsymbol{c}\right] = \frac{1}{Y} \prod_{j=1}^{n} \frac{1}{c_j!} \left(\frac{\mathrm{e}^{-\delta}}{j} \sum_{k \in \mathbb{Z}^d/L} \mathrm{e}^{-j\epsilon(k)} \right)^{c_j} \tag{3.1.5}$$

for $\boldsymbol{c} = (c_j)_{j=1}^n \in \mathbb{N}_0^n$ such that $\sum_{j=1}^n j c_j = n$ and obtain the distribution of the cycle structure.
The key insight in [**11**] is to introduce another measure $\mathbb{P}_n$ which leads to the same distribution of the cycle structure as in Equation (3.1.5). Let $\boldsymbol{n} = (n_k)_{k \in \mathbb{Z}^d/L}$ and $\boldsymbol{\sigma} = (\sigma_k)_{k \in \mathbb{Z}^d/L}$ such that $n_k \in \mathbb{N}_0$ for all k, $\sum_{k \in \mathbb{Z}^d/L} n_k = n$, and $\sigma_k \in S_{n_k}$. We refer to n_k as the occupation number of the Fourier mode k. Moreover, let $\mathcal{N}_{\Lambda,n}$ be the set of pairs $(\boldsymbol{n}, \boldsymbol{\sigma})$ as defined above and define

$$\mathbb{P}_n\left[\{(\boldsymbol{n}, \boldsymbol{\sigma})\}\right] = \frac{1}{Y} \prod_{k \in \mathbb{Z}^d/L} \frac{1}{n_k!} \mathrm{e}^{-\epsilon(k) n_k - \sum_{j \geq 1} \delta C_j(\sigma_k)}.$$

A direct calculation yields that the normalizing constant Y coincides with the one in Equation (3.1.2) and that $\mathbb{P}_{\Lambda,n}\left[\boldsymbol{C} = \boldsymbol{c}\right] = \mathbb{P}_n\left[\boldsymbol{C} = \boldsymbol{c}\right]$ for all $\boldsymbol{c}$. Furthermore, if we now consider $\boldsymbol{n}$ and $\boldsymbol{\sigma}$ as random variables on $\mathcal{N}_{\Lambda,n}$, it can be shown that

$$\mathbb{P}_n\left[\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(0)} \,\middle|\, \boldsymbol{n} = \boldsymbol{n}^{(0)}\right] = \prod_{k \in \mathbb{Z}^d/L} \mathbb{P}_{n_k^{(0)}}^{\left(\mathrm{e}^{-\delta}\right)}\left[\left\{\sigma_k^{(0)}\right\}\right],$$

i.e., given occupation numbers $\boldsymbol{n}^{(0)}$, the permutations σ_k are independent and distributed according to the Ewens measure.
This point is an important step in justifying the intuition regarding the results in Proposition 3.1.1. What remains to be shown is that, in the thermodynamic limit, the occupation numbers divided by the system size, $\frac{n_k}{n}$ for $k \neq 0$, are sufficiently small and that $\frac{n_0}{n}$ converges to ν in a suitable sense. The corresponding properties of the Ewens measure will then entail the convergence of the rescaled longest cycles of σ_0 to PD $\left(\mathrm{e}^{-\delta}\right)$. These steps require involved calculations applying Laplace transforms and, in particular, rely on the Fourier transform of $\mathrm{e}^{-V(x)}$ being non-negative (as does already the definition of $\mathbb{P}_n$). Instead of pursuing this path, we turn to the model of surrogate-spatial random permutations.

3.1.2. Surrogate-Spatial Random Permutations. An important motivation of surrogate-spatial random permutations introduced by Bogachev and Zeindler in [**15**] has been to consider a model approximating that of spatial random permutations which is analytically more tractable. It is defined by non-negative size-dependent cycle weights of the form

$$q_{j,n} = n\kappa_j + \mu_j \tag{3.1.6}$$

for all j and n, so the dependence on n is very explicit. The cycle weights in Equation (3.1.6) are connected with the cycle weights of spatial random permutations given in Equation (3.1.4) by the ansatz

$$\mathrm{e}^{-\delta_j} \sum_{k\in\mathbb{Z}^d/L} \mathrm{e}^{-j\epsilon(k)} = n\kappa_j + \mu_j + o\left(1\right).$$

For instance, in the Gaussian case of $\epsilon\left(k\right) = c\left|k\right|^2$ for some $c > 0$, if j is fixed,

$$\sum_{k\in\mathbb{Z}^d/L} \mathrm{e}^{-j\epsilon(k)} = n\rho^{-1}L^{-d} \sum_{k\in\mathbb{Z}^d/L} \mathrm{e}^{-j\epsilon(k)} \approx n\rho^{-1} \int_{\mathbb{R}^d} \mathrm{e}^{-j\epsilon(v)}\mathrm{d}v$$

is a well-founded approximation in the thermodynamic limit $L, n \to \infty$, and one can justify the choices $\kappa_j = \rho^{-1}\mathrm{e}^{-\delta_j}\int_{\mathbb{R}^d} \mathrm{e}^{-j\epsilon(v)}\mathrm{d}v$ and $\mu_j = 0$. If, however, j grows sufficiently fast with n, it can be shown that appropriate cycle weights are given by $\kappa_j = 0$ and $\mu_j = \mathrm{e}^{-\delta_j}$. Hence, surrogate-spatial random permutations cannot approximate spatial random permutations on all scales simultaneously, but, if calibrated appropriately, they can capture important features of cycles of spatial random permutations in a certain regime. In light of Section 3.1.1, our discussion will focus on the behaviour of long cycles.

The approach in [**15**] rests on the generating functions of the sequences $\left(\frac{\kappa_j}{j}\right)_j$ and $\left(\frac{\mu_j}{j}\right)_j$ given by

$$g_\kappa\left(z\right) = \sum_{j=1}^{\infty} \frac{\kappa_j}{j} z^j \text{ and } g_\mu\left(z\right) = \sum_{j=1}^{\infty} \frac{\mu_j}{j} z^j,$$

respectively. The theorems will be based on analytic assumptions about $g_\kappa\left(z\right)$ and $g_\mu\left(z\right)$ concerning, most importantly, their radii of convergence and their singularities. Assume in the following that $R > 0$ is the radius of convergence of both $g_\kappa\left(z\right)$ and $g_\mu\left(z\right)$. Bogachev and Zeindler distinguish three cases: If $Rg'_\kappa\left(R\right) > 1$, we are in the subcritical regime. The supercritical regime is characterized by $Rg'_\kappa\left(R\right) < 1$ and criticality is given when $Rg'_\kappa\left(R\right) = 1$. We will see that the subcritical and critical cases correspond to $\nu = 0$, whereas the supercritical regime exhibits $\nu > 0$. The distinction between the regimes has analytic reasons which can be glimpsed from considering the relevant normalizing constants given by (cf. Equation (1.3.5))

$$\left[z^n\right]\exp\left(ng_\kappa\left(z\right) + g_\mu\left(z\right)\right) = \frac{1}{2\pi\mathrm{i}}\int_{\partial B_x(0)} \frac{\exp\left(g_\mu\left(z\right)\right)}{z}\exp\left(n\left(g_\kappa\left(z\right) - \log\left(z\right)\right)\right)\mathrm{d}z,$$

where $0 < x < R$. If we want to apply the saddle-point method, the saddle-point equation is

$$0 = g'_\kappa\left(x\right) - \frac{1}{x},$$

or equivalently

$$xg'_\kappa\left(x\right) = 1,$$

which admits a relevant solution if and only if $Rg'_\kappa\left(R\right) \geq 1$.

In a way related to the general approach in this thesis, Bogachev and Zeindler determine the asymptotics of expressions such as

$$\left[z^n\right] f\left(z\right)\exp\left(s\left(ng_\kappa\left(z\right) + g_\mu\left(z\right)\right)\right)$$

in order to extract information about the cycle structure. In the subcritical case, they do so by applying the saddle-point method. In the supercritial regime, they rely on a variation of singularity analysis (see Section 1.3.4). At criticality, depending on concrete assumptions about $g_\kappa\left(z\right)$, one of the two approaches may be applicable.

It is shown that

$$\nu = \max\left\{0, 1 - Rg'_\kappa\left(R\right)\right\}$$

holds, which justifies the nomenclature of the regimes introduced above. While the regime is determined by $g_\kappa\left(z\right)$, the distribution of the longest cycles in the supercritical regime solely depends on $g_\mu(z)$ (see also Proposition 3.1.1): If $g_\mu(z) = -\mu^* \log\left(1-\frac{z}{R}\right) + \mathcal{O}\left(\left(1-\frac{z}{R}\right)^c\right)$ for some $\mu^* > 0$ and $0 < c < 1$ as $z \to R$, i.e. if $g_\mu(z)$ has a logarithmic singularity at $z = R$, then we have

$$\left(\frac{l_1}{\nu n}, \frac{l_2}{\nu n}, \dots\right) \xrightarrow{d} \mathrm{PD}\left(\mu^*\right)$$

as $n \to \infty$. Note that, in particular, $\mu_j = \mu^*$ for all j entails $g_\mu(z) \sim -\mu^* \log(1-z)$ as $z \to 1$.
If, on the other hand, $\mu^* = 0$, i.e. $g_\mu(z)$ does not have a logarithmic singularity at $z = R$, then we have

$$\frac{l_1}{\nu n} \xrightarrow{d} 1$$

as $n \to \infty$. So the model of surrogate-spatial random permutations is capable of replicating the limit distributions in Propositions 3.1.1.
The proximity between the two models becomes even more apparent if an analogue of the density ρ is added to the model of surrogate-spatial random permutations. Define

$$\tilde{\kappa}_j = \tilde{\rho}\kappa_j := \mathrm{e}^{-\delta_j} \int_{\mathbb{R}^d} \mathrm{e}^{-j\epsilon(v)} \mathrm{d}v$$

for $\tilde{\rho} > 0$ and let $g_{\tilde{\kappa}}(z) = \tilde{\rho} g_\kappa(z)$. Then $Rg'_\kappa(R) = \tilde{\rho}^{-1} R g'_{\tilde{\kappa}}(R) > 1$ if and only if $\tilde{\rho} < \tilde{\rho}_c$ for

$$\tilde{\rho}_c = \sum_{j=1}^{\infty} \tilde{\kappa}_j R^j = \sum_{j=1}^{\infty} R^j \mathrm{e}^{-\delta_j} \int_{\mathbb{R}^d} \mathrm{e}^{-j\epsilon(v)} \mathrm{d}v,$$

and we arrive at

$$\nu = \max\left\{0, 1 - \frac{\tilde{\rho}_c}{\tilde{\rho}}\right\}.$$

Among other things, Bogachev and Zeindler also establish limit theorems for the total number of cycles. The corresponding limit distributions are Gaussian expect in one case in the critical regime. It is an interesting open problem to investigate the total number of cycles in spatial random permutations, which would facilitate an additional property with respect to which the two models might be compared.

3.1.3. Spatial Random Permutations and Cycle Weights. In [**23**], Elboim and Peled investigate spatial random permutations (see Section 3.1.1) by adapting and extending the techniques applied by Bogachev and Zeindler in [**15**] to surrogate-spatial permutations to the model at hand. Their assumptions differ from those in [**11**]: They assume that $\mathrm{e}^{-V(x)}$ is a Schwartz function and the probability density of a random variable X such that $\mathbb{E}[X] = 0$. The second assumption is essential, whereas the first one might be weakened to the existence of a certain number of derivatives which decay sufficiently fast. One can give examples which show that the class of functions considered in [**23**] is neither wider nor narrower than the one in [**11**], where, most importantly, Betz and Ueltschi assume non-negativity of the Fourier transform of $\mathrm{e}^{-V(x)}$. Concerning the additional cycle weights δ_j, Elboim and Peled concentrate on the case of constant $\delta_j = \delta \in \mathbb{R}$.
In [**23**], the results strongly depend on the spatial dimension d. Elboim and Peled provide three main theorems which deal with the cases $d = 1$, $d = 2$, and $d \geq 3$, respectively. In each case, they identify subcritical, critical, and supercritical regimes and establish limit theorems for the length of the cycle containing the index 1 (the random variable $\tilde{J}_1$ according to the nomenclature introduced in Section 2.9) in the thermodynamic limit. Instead of the fraction ν of indices contained in infinite cycles, they consider the fraction

$$\tilde{\nu} := \lim_{c \to 0} \liminf_{L, n \to \infty} \mathbb{E}_{\Lambda,n}\left[\frac{1}{n} \sum_{i: l_i \geq cn} l_i\right]$$

of indices contained in macroscopic cycles. Recall that $\mathbb{E}_{\Lambda,n}$ denotes the expectation with respect to the probability measure $\mathbb{P}_{\Lambda,n}$ corresponding to spatial random permutations. Whenever $\tilde{\nu} > 0$ (which is the case in the supercritical regimes and in the critical regime for $d = 1$), they show convergence in distribution of $\left(\frac{l_1}{\tilde{\nu} n}, \frac{l_2}{\tilde{\nu} n}, \dots\right)$ to $\mathrm{PD}\left(\mathrm{e}^{-\delta}\right)$.
Building on the work by Bogachev and Zeindler in [**15**], Elboim and Peled rely on the saddle-point method in the subcritical regime and apply techniques from singularity analysis in the supercritical regime. The results at criticality in dimensions $d \geq 2$ are obtained from those in the subcritical regime by proving a type of monotonicity with respect to increasing the density ρ.

In the supercritical regime in dimension $d \geq 3$, Elboim and Peled prove the statement (1) in Proposition 3.1.1 under their assumptions. In particular, $\tilde{\nu} = \nu$ holds, which is not surprising due to the convergence to the Poisson-Dirichlet distribution. In dimensions $d = 1$ and $d = 2$, the situation is different since Elboim and Peled allow the density $\rho = \rho_n$ to grow (even to diverge) with n in a controlled way in the thermodynamic limit. In particular, no constant finite density $\rho > 0$ is large enough to be outside the subcritical regime if $d \in \{1, 2\}$.
In the following, we discuss one specific result in Theorem 1.2 in [**23**] since it deals with typical cycles of algebraic lengths as does this thesis and additionally elucidates the difference between ν and $\tilde{\nu}$. Recall that $\tilde{J}_1$ denotes the length of the cycle containing the index 1 and let $d = 2$. Let further

$$r_c := \frac{\mathrm{e}^{-\delta}}{2\pi\sqrt{\det\left(\mathrm{Cov}\left(X\right)\right)}},$$

where $\delta \in \mathbb{R}$ is a weight (cf. Equation (3.1.1)) and $\det\left(\mathrm{Cov}\left(X\right)\right)$ denotes the determinant of the covariance matrix of the random variable X with density $\mathrm{e}^{-V(x)}$. We consider the second subcritical regime (the first one being given by constant finite $\rho > 0$) and the critical regime in the theorem: Suppose that $\rho = \rho_n \to \infty$ and $\frac{\rho}{\log(n)} \xrightarrow{n\to\infty} r \in [0, r_c]$ hold. Let U be uniformly distributed on the unit interval $[0, 1]$. Then we have $\tilde{\nu} = 0$ and

$$\frac{r_c \log\left(\tilde{J}_1\right)}{\rho} \xrightarrow{d} U \tag{3.1.7}$$

as $n \to \infty$. By symmetry of the indices, we have

$$\begin{aligned}
\nu &= \lim_{K\to\infty} \liminf_{L,n\to\infty} \mathbb{E}_{\Lambda,n}\left[\frac{1}{n}\sum_{j=1}^{n} \mathbb{1}_{\{\tilde{J}_j \geq K\}}\right] \\
&= \lim_{K\to\infty} \liminf_{L,n\to\infty} \mathbb{E}_{\Lambda,n}\left[\mathbb{1}_{\{\tilde{J}_1 \geq K\}}\right] \\
&= \lim_{K\to\infty} \liminf_{L,n\to\infty} \mathbb{P}_{\Lambda,n}\left[\tilde{J}_1 \geq K\right],
\end{aligned}$$

so Equation (3.1.7) entails $1 = \nu \neq \tilde{\nu}$. Intuitively, in the limit, a typical index belongs to a cycle of infinite length, but not to a macroscopic cycle. If $r > 0$, then $\tilde{J}_1$ lives asymptotically on an algebraic scale. Note, however, that the limiting behaviour of $\tilde{J}_1$ under $\mathbb{P}_{\Lambda,n}$ in this regime is quite different from that of $\tilde{J}_1$ under $\mathbb{P}^{(\vartheta)}_{n,\alpha}$ (see Proposition 2.9.1): Whereas, asymptotically, $\frac{\log\left(\tilde{J}_1\right)}{\log(n)}$ under $\mathbb{P}_{\Lambda,n}$ is uniformly distributed on $\left[0, \frac{r}{r_c}\right]$, i.e. $\tilde{J}_1$ may grow like any power less than $\frac{r}{r_c}$ of n, $\tilde{J}_1$ under $\mathbb{P}^{(\vartheta)}_{n,\alpha}$ grows asymptotically like $\alpha\left(n\right)$.
Concerning the total number of cycles in spatial random permutations, it should be noted that Elboim and Peled outline a way in which the question might be approached, but they do not prove limit theorems.

3.2. Beyond Random Permutations without Macroscopic Cycles

In this section, we discuss three possible directions for generalizing or modifying the model of random permutations without macroscopic cycles in such a way that similar methods might still apply. Firstly, one might choose to allow more general sequences of maximal cycle lengths α which need not grow algebraically in n. A second possibility is considering different cycle weights ϑ_j for cycles of different lengths j or cycle weights which depend on the system size n. A third option consists in dealing with combinatorial structures other than permutations.

Observe that the strategy of many of the proofs in this thesis involves two steps: First, one applies the saddle-point method in the form of Proposition 2.1.4 to a function arising from analytic combinatorics. Then one needs to control the resulting terms in such a way that it is possible to extract the information in question. Any discussion of a generalization of the model which aims to employ similar methods must therefore address these two aspects in the proofs.

3.2.1. More General Sequences α. For the sake of simplicity, we only consider $\vartheta = 1$ in the following discussion of allowing more general sequences α. The assumption in Equation (2.0.2), which underlies most results of the present thesis, can be weakened in certain respects. There are, however, caveats which apply.

Concerning the application of the saddle-point method, one should note that [**45**, Theorem 2] assumes $\alpha(n) \leq \left(12\pi^2 \mathrm{e}\right)^{-1} n (\log(n))^{-1} (\log\log(n))^{-2}$, which is the current state of the art. One should in any case not expect to be able to move beyond the threshold of $\alpha(n) = o(n)$ (see also the relative error term in the statement of the theorem). Theorem 2.6.1 shows that it is indeed possible to arrive at meaningful results for such a wide range of sequences α. Yet, weak assumptions about α also impose limits on the possible perturbations when applying the saddle-point method (condition (2) in Definition 2.1.2 rests indirectly on Equation (2.0.2) and the role played by condition (3) strongly depends on the asymptotics of α).

More importantly, though, the interpretation of the terms provided by the saddle-point method becomes much more difficult and their behaviour changes according to the asymptotics of $\alpha(n)$. Note first that the principal lower bound for $\alpha(n)$ appears to be $\alpha(n) \geq 4$ (see again Theorem 2.6.1). An indication of what has to be taken into account is the behaviour of the saddle point: Lemma 2.2.1 shows that $x_{n,\alpha}(n/\alpha(n))$ diverges if $\alpha(n) = o(\log(n))$, stays finite if $\alpha(n) \approx \log(n)$ and converges to 0 if $\log(n) = o(\alpha(n))$. A central problem is the fact that we only control the asymptotics of $(x_{n,\alpha}(n/\alpha(n)))^{\alpha(n)}$, which is a term of vital importance in many calculations (e.g., asymptotic expected values and limit shape), if $\alpha(n)$ grows at least logarithmically. But the approach which works for algebraic α often fails even when $\log(n) = o(\alpha(n))$ still holds. Consider, for instance, that the proofs of the general case in Section 2.6 are more complex and provide weaker results (compare, e.g., Proposition 2.6.6 and Theorem 2.6.3) since certain approximations in the proof of the special case lead to error terms unwieldy for general α.

The underlying fact that constrained random permutations exhibit a variety of behaviour depending on the maximal cycle length α (most indices concentrated in cycles of maximal length if α is bounded by a small number and almost classical behaviour in certain respects for fast-growing sequences α) is thus reflected in analytic problems specific to the asymptotic behaviour of α. The strategy in Section 2.6, which tries to deal in a uniform way with a wide range of sequences α, is only feasible because the central limit theorem for the total number of cycles is a phenomenon very stable with respect to α. A unified approach, however, will have to give way to a plurality of methods geared towards more specific situations when applied to problems such as the limit shape, which, e.g., cannot be expected to hold for $\alpha(n) = o(\log(n))$. The most interesting regime in this respect concerns sequences α which increase slowly in n and its study requires first and foremost determining the asymptotics of $(x_{n,\alpha}(n/\alpha(n)))^{\alpha(n)}$.

3.2.2. More General Cycle Weights. There are many options when it comes to considering cycle weights other than a constant $\vartheta > 0$. For instance, if one thinks of random random A-permutations, it is natural to define size-dependent sets $A_n \subset \{j \in \mathbb{N} : j \leq \alpha(n)\}$ which denote admissible cycle lengths in a permutation of n elements, thereby combining aspects of both random A-permutations and random permutations without macroscopic cycles. Provided that the set $\{j \in \mathbb{N} : j \leq \alpha(n)\} \backslash A_n$ is sufficiently sparse for each n (in particular, there should be no periodicity in A_n), the results of this thesis should be quite stable under this generalization.

A second way of generalizing our model consists in allowing cycle weights ϑ_j which depend non-trivially on the lengths j of the cycles. Since there is a broad variety of models of random permutations with cycle weights (cf. the references given in the introduction), there is an equally rich field of such models conditioned to have no macroscopic cycles. Whether the methods outlined in this thesis still apply to such a model will naturally depend on the specific behaviour of the cycle weights in question.
We therefore focus our discussion on a third option. Recall that Theorem 2.5.1 basically tells us that imposing a maximal cycle length satisfying Equation (2.0.2) on, e.g., uniform random permutations does not affect the asymptotic behaviour of the counts of short cycles. The stability of the behaviour of short cycles is a phenomenon common to many models. If we want to produce a model in which the numbers of finite cycles do not stay bounded for large n, we may define cycle weights $\vartheta = \vartheta(n) = (\log(n))^{N_0}$ for $N_0 \in \mathbb{N}$ which do not depend on the lengths of the cycles, but on the system size n. The given choice of $\vartheta(n)$ will gear the model towards a greater number of cycles, which is in particular going to favour the occurence of more short cycles. Observe further that, by Lemma 2.2.1, the triangular array $\boldsymbol{q}_{\log}$ with $q_{j,n} = (\log(n))^{N_0}$ for $1 \leq j \leq \alpha(n)$ satisfies conditions (1) and (2) in Definition 2.1.1. Since it does not fulfil condition (3), it is strictly speaking not admissible. A look at the relevant part of tails pruning in the proof of Proposition 2.1.4, however, shows that the arguments also work in this case. Hence, we may apply the saddle-point method to the model of random permutations without macroscopic cycles described by the array $\boldsymbol{q}_{\log}$. By following the approach of Proposition 2.3.2, we can determine the asymptotic behaviour of the expected cycle counts. In particular, the expected value of cycles of length 1 will grow like $(\log(n))^{N_0}$. Due to the saddle-point method and since we have control of the saddle point, further analysis of the model with similar methods should be possible.

3.2.3. Other Structures. The results in this thesis pertain only to probabilistic models of permutations. There are, however, many more decomposable structures which fall into the categories of logarithmic combinatorial or, more broadly, combinatorial structures. Depending on the generating functions in question, similar methods might enable a study of such models in which the maximal size of components is also bounded by a sequence α. In a first step, one would have to find a counterpart of the local probability of permutations without macroscopic cycles under the uniform measure developped in [**45**]. In a second step, such a result might lend itself in a similar way to the study of the probabilistic model of the combinatorial structure without macroscopic components.

APPENDIX A

A.1. Further Proofs and Auxiliary Lemmata

A.1.1. Properties of the Derivatives of $h_{n,C}$. We start with showing that the saddle point $x_{n,C}$ is differentiable with respect to s. Let

$$\tilde{\lambda}_{p,n}(x) := \sum_{j=1}^{\alpha(n)} j^{p-1} x^j$$

and $\tilde{\lambda}_{p,n} := \tilde{\lambda}_{p,n}(x_{n,C}(s))$. Since $x_{n,C}(s) = x_{n,\boldsymbol{q}} = x_{n,\alpha}\left(\frac{n}{\alpha(n)\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}\right)$, this is consistent with the definition in Section 2.2. By Lemma 2.2.1, we further conclude that

$$\tilde{\lambda}_{2,n} \sim \frac{n\alpha(n)}{\vartheta} \tag{A.1.1}$$

locally uniformly in $s \geq 0$ if $\alpha(n) = o(n)$ since $\gamma(n)$ tends to infinity.

LEMMA A.1.1 ([**9**, Lemma 4.1]). *For fixed n, the saddle point $x_{n,C}$ is an infinitely often differentiable function in s. In particular,*

$$\frac{x'_{n,C}(s)}{x_{n,C}(s)} \tilde{\lambda}_{2,n} = -\frac{n}{\gamma(n)} \frac{\mathrm{e}^{-\frac{s}{\gamma(n)}}}{\vartheta}. \tag{A.1.2}$$

PROOF. Fix a number n and consider the function

$$g(x) := \sum_{j=1}^{\alpha(n)} x^j.$$

One easily sees that g is strictly increasing on $[0, \infty)$ and that $g'(x) > 0$ holds for all $x > 0$. By Equation (2.6.9), we have

$$x_{n,C}(s) = g^{-1}\left(\frac{n}{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}\right).$$

Since g^{-1} is differentiable by the inverse mapping theorem, we conclude that $x_{n,C}$ is also differentiable. More precisely, the first derivative of $x_{n,C}$ is given by

$$x'_{n,C}(s) = -\frac{1}{g'(x_{n,C}(s))} \frac{n\mathrm{e}^{-\frac{s}{\gamma(n)}}}{\vartheta\gamma(n)} \tag{A.1.3}$$

and therefore also differentiable. More generally, a bootstrapping argument shows that $x_{n,C}$ is indeed infinitely often differentiable. Equation (A.1.2) follows from Equation (A.1.3) and the definition of $\tilde{\lambda}_{2,n}$. □

Differentiability of $x_{n,C}$ entails differentiability of $h_{n,C}$, so we can start calculating the derivatives of $h_{n,C}$.

LEMMA A.1.2 ([**9**, Lemma 4.4]). *Let $\alpha : \mathbb{N} \to \mathbb{N}$ such that $n/(\vartheta\alpha(n)) > 3$ for large n and*

$$\lim_{n\to\infty} \frac{\alpha(n)}{n} = 0.$$

Then, locally uniformly in $s \geq 0$,

$$\frac{x'_{n,C}(s)}{x_{n,C}(s)} \sim -\frac{1}{\alpha(n)\gamma(n)} \tag{A.1.4}$$

and

$$\frac{x''_{n,C}(s)}{x_{n,C}(s)} = \mathcal{O}\left(\frac{1}{\alpha(n)(\gamma(n))^2}\right) \tag{A.1.5}$$

as $n \to \infty$.

PROOF. By Lemma A.1.1, we have

$$\frac{x'_{n,C}(s)}{x_{n,C}(s)}\tilde{\lambda}_{2,n} = -\frac{n}{\gamma(n)}\frac{\mathrm{e}^{-s/\gamma(n)}}{\vartheta}.$$

Equation (A.1.4) is therefore a direct consequence of Equation (A.1.1) and $\gamma(n) \to \infty$ as $n \to \infty$. If we differentiate once more, we obtain

$$\frac{x''_{n,C}(s)}{x_{n,C}(s)}\tilde{\lambda}_{2,n} - \left(\frac{x'_{n,C}(s)}{x_{n,C}(s)}\right)^2\tilde{\lambda}_{2,n} + \left(\frac{x'_{n,C}(s)}{x_{n,C}(s)}\right)^2\tilde{\lambda}_{3,n} = \frac{n}{(\gamma(n))^2}\frac{\mathrm{e}^{-s/\gamma(n)}}{\vartheta},$$

from which the claim follows by applying Equations (A.1.1) and (A.1.4). □

LEMMA A.1.3. *Let* $\alpha : \mathbb{N} \to \mathbb{N}$ *as in Lemma A.1.2. Then the following relations hold:*

$$h'_{n,C}(0) = \frac{1}{\gamma(n)}\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}x_{n,\vartheta}^j, \tag{A.1.6}$$

$$h''_{n,C}(0) = \frac{1}{(\gamma(n))^2}\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}x_{n,\vartheta}^j + \frac{n}{\gamma(n)}\frac{x'_{n,C}(0)}{x_{n,\vartheta}} \tag{A.1.7}$$

$$= \frac{1}{(\gamma(n))^2}\sum_{j=1}^{\alpha(n)}\frac{\vartheta}{j}x_{n,\vartheta}^j - \frac{n^2}{(\gamma(n))^2}\frac{1}{\sum_{j=1}^{\alpha(n)} j\vartheta x_{n,\vartheta}^j}, \text{ and} \tag{A.1.8}$$

$$h'''_{n,C}(s) = \frac{1}{(\gamma(n))^3}\sum_{j=1}^{\alpha(n)}\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{j}(x_{n,C}(s))^j + \frac{n}{(\gamma(n))^2}\frac{x'_{n,C}(s)}{x_{n,C}(s)} + \frac{n}{\gamma(n)}\left(\frac{x''_{n,C}(s)}{x_{n,C}(s)} - \left(\frac{x'_{n,C}(s)}{x_{n,C}(s)}\right)^2\right). \tag{A.1.9}$$

PROOF. Recall that

$$h_{n,C}(s) = \sum_{j=1}^{\alpha(n)}\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{j}(x_{n,C}(s))^j - n\log(x_{n,C}(s))$$

by Equation (2.6.10). Hence,

$$\begin{aligned} h'_{n,C}(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\sum_{j=1}^{\alpha(n)}\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{j}(x_{n,C}(s))^j - n\log(x_{n,C}(s))\right] \\ &= \frac{1}{\gamma(n)}\sum_{j=1}^{\alpha(n)}\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{j}(x_{n,C}(s))^j + \frac{x'_{n,C}(s)}{x_{n,C}(s)}\sum_{j=1}^{\alpha(n)}\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}(x_{n,C}(s))^j - n\frac{x'_{n,C}(s)}{x_{n,C}(s)} \\ &= \frac{1}{\gamma(n)}\sum_{j=1}^{\alpha(n)}\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{j}(x_{n,C}(s))^j \end{aligned}$$

by Equation (2.6.9). So Equation (A.1.6) follows from the definition of $x_{n,\vartheta}$. Further differentiating yields

$$\begin{aligned} h''_{n,C}(s) =& \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\gamma(n)}\sum_{j=1}^{\alpha(n)}\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{j}\left(x_{n,C}(s)\right)^j\right] \\ =& \frac{1}{(\gamma(n))^2}\sum_{j=1}^{\alpha(n)}\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{j}\left(x_{n,C}(s)\right)^j + \frac{1}{\gamma(n)}\frac{x'_{n,C}(s)}{x_{n,C}(s)}\sum_{j=1}^{\alpha(n)}\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}\left(x_{n,C}(s)\right)^j \\ =& \frac{1}{(\gamma(n))^2}\sum_{j=1}^{\alpha(n)}\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{j}\left(x_{n,C}(s)\right)^j + \frac{n}{\gamma(n)}\frac{x'_{n,C}(s)}{x_{n,C}(s)}, \end{aligned}$$

which entails Equation (A.1.7). Equation (A.1.8) is then a consequence of Lemma A.1.1. Furthermore,

$$\begin{aligned} h'''_{n,C}(s) =& \frac{1}{(\gamma(n))^3}\sum_{j=1}^{\alpha(n)}\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{j}\left(x_{n,C}(s)\right)^j + \frac{1}{(\gamma(n))^2}\sum_{j=1}^{\alpha(n)}\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}\left(x_{n,C}(s)\right)^j\frac{x'_{n,C}(s)}{x_{n,C}(s)} \\ &+ \frac{n}{\gamma(n)}\left(\frac{x''_{n,C}(s)}{x_{n,C}(s)} - \left(\frac{x'_{n,C}(s)}{x_{n,C}(s)}\right)^2\right) \\ =& \frac{1}{(\gamma(n))^3}\sum_{j=1}^{\alpha(n)}\frac{\vartheta \mathrm{e}^{\frac{s}{\gamma(n)}}}{j}\left(x_{n,C}(s)\right)^j + \frac{n}{(\gamma(n))^2}\frac{x'_{n,C}(s)}{x_{n,C}(s)} + \frac{n}{\gamma(n)}\left(\frac{x''_{n,C}(s)}{x_{n,C}(s)} - \left(\frac{x'_{n,C}(s)}{x_{n,C}(s)}\right)^2\right). \end{aligned}$$

Here, the second line applies Equation (2.6.9). The claim is proved. □

LEMMA A.1.4. *Let $\alpha : \mathbb{N} \to \mathbb{N}$ as in Lemma A.1.2. Then, locally uniformly in $s \geq 0$,*

$$h'''_{n,C}(s) = \mathcal{O}\left(\frac{h''_n(0)}{\gamma(n)}\right) + \mathcal{O}\left(\frac{n}{\alpha(n)(\gamma(n))^3}\right).$$

PROOF. By Lemma A.1.3 and due to

$$h''_{n,C}(s) = \mathcal{O}\left(h''_{n,C}(0)\right),$$

locally uniformly in $s \geq 0$, we only have to show that

$$\frac{n}{\gamma(n)}\left(\frac{x''_{n,C}(s)}{x_{n,C}(s)} - \left(\frac{x'_{n,C}(s)}{x_{n,C}(s)}\right)^2\right) = \mathcal{O}\left(\frac{n}{\alpha(n)(\gamma(n))^3}\right)$$

as n tends to infinity. By Equations (A.1.4) and (A.1.5),

$$\frac{x''_{n,C}(s)}{x_{n,C}(s)} - \left(\frac{x'_{n,C}(s)}{x_{n,C}(s)}\right)^2 = \mathcal{O}\left(\frac{1}{\alpha(n)(\gamma(n))^2} + \frac{1}{(\alpha(n)\gamma(n))^2}\right)$$

holds locally uniformly in s, and the claim follows. □

A.1.2. Properties of the Derivatives of $h_{n,K}$ and $h_{n,S}$. In this section we prove the properties of $h_{n,K}$ and $h_{n,S}$ which are applied in Sections 2.7.2 and 2.7.3. Recall that the functions are defined by

$$h_{n,K}(\boldsymbol{s}) = \sum_{k=0}^{M}\mathrm{e}^{\sum_{i=k+1}^{M}\frac{s_i}{\gamma(n)}}\sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)}\frac{\vartheta}{j}\left(x_{n,K}(\boldsymbol{s})\right)^j - n\log\left(x_{n,K}(\boldsymbol{s})\right)$$

and

$$h_{n,S}(\boldsymbol{s}) = \sum_{k=0}^{M}\sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)}\frac{\vartheta}{j}\left(\mathrm{e}^{\sum_{i=k+1}^{M}\frac{s_i}{\gamma(n)}}x_{n,S}(\boldsymbol{s})\right)^j - n\log\left(x_{n,S}(\boldsymbol{s})\right),$$

for given $\boldsymbol{t} = (t_k)_{k=1}^{M}$ such that $0 = t_0 \leq t_1 < t_2 < \cdots < t_M \leq t_{M+1} = 1$ and $(\gamma(n))_{n\in\mathbb{N}}$ such that $\gamma(n) = \Omega(\log(n))$. We will prove the properties for both functions simultaneously since $h_{n,K}$ and

$h_{n,S}$ exhibit a similar structure. The first property states that $h_{n,K}$ and $h_{n,S}$ are infinitely often differentiable. It is a consequence of differentiability of the saddle points.

LEMMA A.1.5 ([**10**, Lemma 4.9]). *The saddle points $x_{n,K}$ and $x_{n,S}$ are infinitely often differentiable with respect to $\boldsymbol{s}$.*

PROOF. Consider, for $x > 0$, the functions

$$F_1(\boldsymbol{s}, x) := \sum_{k=0}^{M} \mathrm{e}^{\sum_{i=k+1}^{M} \frac{s_i}{\gamma(n)}} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta x^j - n$$

and

$$F_2(\boldsymbol{s}, x) := \sum_{k=0}^{M} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta \left(\mathrm{e}^{\sum_{i=k+1}^{M} \frac{s_i}{\gamma(n)}} x\right)^j - n,$$

which are motivated by Equations (2.7.5) and (2.7.30). One easily sees that both F_1 and F_2 are infinitely often differentiable with respect to $\boldsymbol{s}$ and x. Further, $\partial_x F_1 \neq 0$ and $\partial_x F_2 \neq 0$ for all positive x. Since $0 = F_1(\boldsymbol{s}, x_{n,K}(\boldsymbol{s}))$ and $0 = F_2(\boldsymbol{s}, x_{n,S}(\boldsymbol{s}))$ by the definition of the saddle points, the claim follows from the implicit function theorem (see, e.g., [**41**, Theorem 7.9]). □

It is now possible to calculate the derivatives. For the sake of brevity, we introduce the notations

$$\lambda_{p,n,K}^{(l)} := \sum_{k=0}^{l-1} \mathrm{e}^{\sum_{i=k+1}^{M} \frac{s_i}{\gamma(n)}} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta j^{p-1} \left(x_{n,K}(\boldsymbol{s})\right)^j$$

and

$$\lambda_{p,n,K} := \lambda_{p,n,K}^{(M+1)}$$

as well as

$$\lambda_{p,n,S}^{(l)} := \sum_{k=0}^{l-1} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta j^{p-1} \left(\mathrm{e}^{\sum_{i=k+1}^{M} \frac{s_i}{\gamma(n)}} x_{n,S}(\boldsymbol{s})\right)^j$$

and

$$\lambda_{2,n,S} := \lambda_{2,n,S}^{(M+1)}.$$

Fix $k_3 \leq k_2 \leq k_1$. Then we have

$$\begin{aligned}
\partial_{s_{k_1}} h_{n,K}(\boldsymbol{s}) =& \frac{1}{\gamma(n)} \sum_{k=0}^{k_1-1} \mathrm{e}^{\sum_{i=k+1}^{M} \frac{s_i}{\gamma(n)}} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \frac{\vartheta}{j} \left(x_{n,K}(\boldsymbol{s})\right)^j \\
&+ \frac{\partial_{s_{k_1}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} \left[\sum_{k=0}^{M} \mathrm{e}^{\sum_{i=k+1}^{M} \frac{s_i}{\gamma(n)}} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta \left(x_{n,K}(\boldsymbol{s})\right)^j - n\right] \\
=& \frac{1}{\gamma(n)} \lambda_{0,n,K}^{(k_1)}, \qquad \text{(A.1.10)}
\end{aligned}$$

where the last line follows from Equation (2.7.5). A similar calculation yields

$$\begin{aligned}
\partial_{s_{k_1}} h_{n,S}(\boldsymbol{s}) =& \frac{1}{\gamma(n)} \sum_{k=0}^{k_1-1} \sum_{j=b_{t_k}(n)+1}^{b_{t_{k+1}}(n)} \vartheta \left(\mathrm{e}^{\sum_{i=k+1}^{M} \frac{s_i}{\gamma(n)}} x_{n,S}(\boldsymbol{s})\right)^j \\
=& \frac{1}{\gamma(n)} \lambda_{1,n,S}^{(k_1)}. \qquad \text{(A.1.11)}
\end{aligned}$$

Hence,

$$\partial_{s_{k_2}} \partial_{s_{k_1}} h_{n,K}(\boldsymbol{s}) = \frac{1}{\gamma(n)} \frac{\partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} \lambda_{1,n,K}^{(k_1)} + \frac{1}{(\gamma(n))^2} \lambda_{0,n,K}^{(k_2)} \qquad \text{(A.1.12)}$$

and

$$\partial_{s_{k_2}} \partial_{s_{k_1}} h_{n,S}(\boldsymbol{s}) = \frac{1}{\gamma(n)} \frac{\partial_{s_{k_2}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} \lambda_{2,n,S}^{(k_1)} + + \frac{1}{(\gamma(n))^2} \lambda_{2,n,S}^{(k_2)} \qquad \text{(A.1.13)}$$

hold. Since we have

$$\partial_{s_{k_3}} \frac{\partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} = \frac{\partial_{s_{k_3}} \partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} - \frac{\partial_{s_{k_3}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} \frac{\partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})}$$

(and analogously for $x_{n,S}(\boldsymbol{s})$) by the quotient rule, we obtain

$$\begin{aligned}
&\partial_{s_{k_3}} \partial_{s_{k_2}} \partial_{s_{k_1}} h_{n,K}(\boldsymbol{s}) \\
=& \frac{1}{(\gamma(n))^2} \frac{\partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} \lambda_{1,n,K}^{(k_3)} \\
&+ \frac{1}{\gamma(n)} \left(\frac{\partial_{s_{k_3}} \partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} - \frac{\partial_{s_{k_3}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} \frac{\partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} \right) \lambda_{1,n,K}^{(k_1)} \\
&+ \frac{1}{\gamma(n)} \frac{\partial_{s_{k_3}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} \frac{\partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} \lambda_{2,n,K}^{(k_1)} + \frac{1}{(\gamma(n))^3} \lambda_{0,n,K}^{(k_3)} \\
&+ \frac{1}{(\gamma(n))^2} \frac{\partial_{s_{k_3}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} \lambda_{1,n,K}^{(k_2)}
\end{aligned} \tag{A.1.14}$$

and

$$\begin{aligned}
&\partial_{s_{k_3}} \partial_{s_{k_2}} \partial_{s_{k_1}} h_{n,S}(\boldsymbol{s}) \\
=& \frac{1}{(\gamma(n))^2} \frac{\partial_{s_{k_2}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} \lambda_{3,n,S}^{(k_3)} \\
&+ \frac{1}{\gamma(n)} \left(\frac{\partial_{s_{k_3}} \partial_{s_{k_2}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} - \frac{\partial_{s_{k_3}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} \frac{\partial_{s_{k_2}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} \right) \lambda_{2,n,S}^{(k_1)} \\
&+ \frac{1}{\gamma(n)} \frac{\partial_{s_{k_3}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} \frac{\partial_{s_{k_2}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} \lambda_{3,n,S}^{(k_1)} + \frac{1}{(\gamma(n))^3} \lambda_{3,n,S}^{(k_3)} \\
&+ \frac{1}{(\gamma(n))^2} \frac{\partial_{s_{k_3}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} \lambda_{3,n,S}^{(k_2)}
\end{aligned} \tag{A.1.15}$$

for the third derivatives.
The next step is deriving asymptotics of the derivatives of the saddle points in question.

LEMMA A.1.6 ([**10**, Lemma 4.10]). *Under the assumptions of Lemma 2.7.12, we have for $k_2 \leq k_1$ that*

$$\frac{\partial_{s_{k_1}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} = -\frac{1}{\gamma(n)} \frac{\lambda_{1,n,K}^{(k_1)}}{\lambda_{2,n,K}}. \tag{A.1.16}$$

Moreover,

$$\frac{\partial_{s_{k_1}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} = \mathcal{O}\left(\frac{1}{\gamma(n)\,\alpha(n)}\right) \tag{A.1.17}$$

and

$$\frac{\partial_{s_{k_2}} \partial_{s_{k_1}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} - \frac{\partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} \frac{\partial_{s_{k_1}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} = \mathcal{O}\left(\frac{1}{(\gamma(n))^2\,\alpha(n)}\right) \tag{A.1.18}$$

hold locally uniformly in $\boldsymbol{s}$.

PROOF. If we differentiate Equation (2.7.5) with respect to s_{k_1}, we obtain

$$0 = \frac{1}{\gamma(n)} \lambda_{1,n,K}^{(k_1)} + \frac{\partial_{s_{k_1}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} \lambda_{2,n,K}.$$

Equivalently,

$$\frac{\partial_{s_{k_1}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} = -\frac{\frac{1}{\gamma(n)} \lambda_{1,n,K}^{(k_1)}}{\lambda_{2,n,K}},$$

which proves Equation (A.1.16). By Equation (2.7.5), we have

$$\lambda_{1,n,K}^{(k_1)} \leq n.$$

Lemma 2.7.13 states that

$$\lambda_{2,n,K} \sim n\alpha(n)$$

locally uniformly in $\boldsymbol{s}$. Hence,

$$\frac{\partial_{s_{k_1}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} = \mathcal{O}\left(\frac{1}{\gamma(n)\,\alpha(n)}\right)$$

follows, and Equation (A.1.17) is proved. By differentiating Equation (A.1.16) with respect to s_{k_2}, we arrive at

$$\begin{aligned}
&\frac{\partial_{s_{k_2}}\partial_{s_{k_1}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})} - \frac{\partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})}\frac{\partial_{s_{k_1}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})}\\
=& -\frac{1}{(\gamma(n))^2}\frac{\lambda^{(k_2)}_{1,n,K}}{\lambda_{2,n,K}} - \frac{1}{\gamma(n)}\frac{\partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})}\frac{\lambda^{(k_1)}_{2,n,K}}{\lambda_{2,n,K}}\\
&+\frac{1}{(\gamma(n))^2}\frac{\lambda^{(k_1)}_{1,n,K}}{(\lambda_{2,n,K})^2}\lambda^{(k_2)}_{2,n,K} + \frac{1}{\gamma(n)}\frac{\lambda^{(k_1)}_{1,n,K}}{(\lambda_{2,n,K})^2}\frac{\partial_{s_{k_2}} x_{n,K}(\boldsymbol{s})}{x_{n,K}(\boldsymbol{s})}\lambda_{3,n,K}.
\end{aligned}$$

Equation (A.1.18) then follows from applying Lemma 2.7.13 and $j \le \alpha(n)$ as well as Equations (A.1.17) and (2.7.5) to each term individually. □

The second property of $h_{n,K}$ stated in Section 2.7.2 is a consequence of Equation (A.1.10) and Lemma 2.7.14. Properties (iii), (iv), and (v) now follow from Equations (A.1.12) and (A.1.14) as well as Lemmata A.1.6 and 2.7.14.

LEMMA A.1.7. *Under the assumptions of Lemma 2.7.18, we have for $k_2 \le k_1$ that*

$$\frac{\partial_{s_{k_1}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} = -\frac{1}{\gamma(n)}\frac{\lambda^{(k_1)}_{2,n,S}}{\lambda_{2,n,S}}. \tag{A.1.19}$$

Furthermore,

$$\frac{\partial_{s_{k_1}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} = \mathcal{O}\left(\frac{1}{\gamma(n)}\right) \tag{A.1.20}$$

and

$$\frac{\partial_{s_{k_2}}\partial_{s_{k_1}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} - \frac{\partial_{s_{k_2}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})}\frac{\partial_{s_{k_1}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} = \mathcal{O}\left(\frac{\alpha(n)}{(\gamma(n))^2}\right) \tag{A.1.21}$$

hold locally uniformly in $\boldsymbol{s}$.

PROOF. The proof follows the same approach as the proof of Lemma A.1.6. Differentiating Equation (2.7.30) with respect to s_{k_1} yields

$$0 = \frac{1}{\gamma(n)}\lambda^{(k_1)}_{2,n,S} + \frac{\partial_{s_{k_1}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})}\lambda_{2,n,S}.$$

So Equation (A.1.19) is proved. Equation (A.1.20) is then a consequence of Lemma 2.7.18, $j \le \alpha(n)$, and Equation (2.7.30). If we differentiate Equation (A.1.19) with respect to s_{k_2},

$$\begin{aligned}
&\frac{\partial_{s_{k_2}}\partial_{s_{k_1}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})} - \frac{\partial_{s_{k_2}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})}\frac{\partial_{s_{k_1}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})}\\
=& -\frac{1}{(\gamma(n))^2}\frac{\lambda^{(k_2)}_{3,n,S}}{\lambda_{2,n,S}} - \frac{1}{\gamma(n)}\frac{\partial_{s_{k_2}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})}\frac{\lambda^{(k_1)}_{3,n,S}}{\lambda_{2,n,S}}\\
&+\frac{1}{(\gamma(n))^2}\frac{\lambda^{(k_1)}_{2,n,S}}{(\lambda_{2,n,S})^2}\lambda^{(k_2)}_{3,n,S} + \frac{1}{\gamma(n)}\frac{\lambda^{(k_1)}_{2,n,S}}{(\lambda_{2,n,S})^2}\frac{\partial_{s_{k_2}} x_{n,S}(\boldsymbol{s})}{x_{n,S}(\boldsymbol{s})}\lambda_{3,n,S}
\end{aligned}$$

follows. Equation (A.1.21) can then be proved by considering each term individually and applying Lemma 2.7.18 and $j \le \alpha(n)$ as well as Equations (A.1.20) and (2.7.30). □

Property (ii) of $h_{n,S}$ stated in Section 2.7.3 follows from Equation (A.1.11) and Lemma 2.7.14. Properties (iii) and (iv) can be deduced from Equations (A.1.13) and (A.1.15) as well as Lemmata A.1.7 and 2.7.14.

A.1.3. Proof of Lemma 2.7.15. Recall that

$$c_t(n) := \begin{cases} \left\lfloor \alpha(n) + \log(t) \frac{\alpha(n)}{\log\left(\frac{n}{\alpha(n)}\right)} \right\rfloor & \text{if } \frac{1}{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2} \le t \le 1 \\ \left\lfloor t \cdot \left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2 \alpha(n) \left(1 - 2\frac{\log\log\left(\frac{n}{\alpha(n)}\right)}{\log\left(\frac{n}{\alpha(n)}\right)}\right) \right\rfloor & \text{if } 0 \le t < \frac{1}{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2} \end{cases}$$

for $t \in [0,1]$ and $n \in \mathbb{N}$. Since the behaviour of $\vartheta \frac{x_{n,\vartheta}^j}{j}$ strongly depends on j, we will first prove Lemma A.1.8 below which deals with t_1 and t_2 such that $c_{t_2}(n) - c_{t_1}(n)$ is bounded from above by a constant. The claim of Lemma 2.7.15 for more general t_1 and t_2 then follows from finding suitable points $t_1 < \tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_l < t_2$ to which Lemma A.1.8 can be applied in the following way:

$$\begin{aligned} \sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j &= \sum_{j=c_{t_1}(n)+1}^{c_{\tilde{t}_1}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j + \sum_{i=1}^{l-1} \sum_{j=c_{\tilde{t}_i}(n)+1}^{c_{\tilde{t}_{i+1}}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j + \sum_{j=c_{\tilde{t}_l}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \\ &\le C \frac{n}{\alpha(n)} \left[\left(\tilde{t}_1^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right) + \sum_{i=1}^{l-1} \left(\tilde{t}_{i+1}^{\frac{1}{r}} - \tilde{t}_i^{\frac{1}{r}}\right) + \left(t_2^{\frac{1}{r}} - \tilde{t}_l^{\frac{1}{r}}\right) \right] \\ &\le C \frac{n}{\alpha(n)} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right). \end{aligned}$$

In order to prove Lemma A.1.8, we collect a few useful inequalities. Due to Lemma 2.3.1, we have

$$x_{n,\vartheta}^{\alpha(n)} \le 2 \exp\left(\log\left(\frac{n}{\alpha(n)\vartheta} \log\left(\frac{n}{\alpha(n)\vartheta}\right)\right)\right) \tag{A.1.22}$$

and

$$\log\left(\frac{n}{\alpha(n)\vartheta} \log\left(\frac{n}{\alpha(n)\vartheta}\right)\right) \ge \log\left(\frac{n}{\alpha(n)}\right) \tag{A.1.23}$$

for large n. Therefore, if $b_t(n) > 0$ holds, we obtain

$$\begin{aligned} x_{n,\vartheta}^{b_t(n)} &= \left(x_{n,\vartheta}^{\alpha(n)}\right)^{\frac{b_t(n)}{\alpha(n)}} \\ &\le 2 \exp\left(\frac{b_t(n)}{\alpha(n)} \log\left(\frac{n}{\alpha(n)\vartheta} \log\left(\frac{n}{\alpha(n)\vartheta}\right)\right)\right) \\ &\le 2 \frac{n}{\alpha(n)\vartheta} \log\left(\frac{n}{\alpha(n)\vartheta}\right) \exp\left(\log(t) \frac{\log\left(\frac{n}{\alpha(n)\vartheta} \log\left(\frac{n}{\alpha(n)\vartheta}\right)\right)}{\log\left(\frac{n}{\alpha(n)}\right)}\right) \\ &\le 2 \frac{n}{\alpha(n)\vartheta} \log\left(\frac{n}{\alpha(n)\vartheta}\right) t. \end{aligned} \tag{A.1.24}$$

Here, the fourth line follows from $0 < t \le 1$.

LEMMA A.1.8. *Let $N \in \mathbb{N}$ and $c > 0$ such that*

$$\frac{1}{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2} + \frac{1}{\alpha(n) \log\left(\frac{n}{\alpha(n)}\right)} \le c \frac{1}{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2} \tag{A.1.25}$$

and

$$b_{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2}}(n) \ge \frac{\alpha(n)}{2} \tag{A.1.26}$$

as well as Equations (A.1.22) and (A.1.23) hold for all $n \ge N$. Let further $r \ge 3$ large enough so that $a_2\left(1 + \frac{1}{r}\right) < 1$, where a_2 is defined in Equation (2.0.2). Then there is a constant $C > 0$ such that

$$\sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \le C \frac{n}{\alpha(n)} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right)$$

for all $n \ge N$ and $0 \le t_1 < t_2 \le 1$ satisfying $10 \ge c_{t_2}(n) - c_{t_1}(n) \ge 2$.

PROOF. In the following, C is used to denote a generic positive constant. Its value may change from line to line.
We have

$$
\begin{aligned}
\sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^{j} &= \sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} + \vartheta \int_1^{x_{n,\vartheta}} \sum_{j=c_{t_1}(n)}^{c_{t_2}(n)-1} v^j \mathrm{d}v \\
&\leq \sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} + \vartheta \sum_{j=0}^{c_{t_2}(n)-c_{t_1}(n)-1} x_{n,\vartheta}^{j} \int_1^{x_{n,\vartheta}} v^{c_{t_1}(n)} \mathrm{d}v \\
&= \sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} + \vartheta \frac{x_{n,\vartheta}^{c_{t_2}(n)-c_{t_1}(n)} - 1}{x_{n,\vartheta} - 1} \int_1^{x_{n,\vartheta}} v^{c_{t_1}(n)} \mathrm{d}v \\
&= \sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} + \vartheta \frac{x_{n,\vartheta}^{c_{t_2}(n)-c_{t_1}(n)} - 1}{x_{n,\vartheta} - 1} \frac{x_{n,\vartheta}^{c_{t_1}(n)+1} - 1}{c_{t_1}(n) + 1}.
\end{aligned} \tag{A.1.27}
$$

Step 1: Assume first that $t_1, t_2 \leq \left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2}$.
In the following, we will deal with both terms in Equation (A.1.27) separately. Assume w.l.o.g. that $c_{t_1}(n) > 0$ (otherwise work with $c_{\tilde{t}}(n)$ where $\tilde{t}$ is the smallest t such that $c_t(n) = 1$ and apply $t_2^{\frac{1}{r}} - \tilde{t}^{\frac{1}{r}} \leq t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}$), which entails $t_1 \geq \frac{1}{\alpha(n)\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2}$. Hence,

$$
\begin{aligned}
\sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} \leq &\frac{\vartheta}{c_{t_1}(n)+1}\left(c_{t_2}(n) - c_{t_1}(n)\right) \\
\leq &C \frac{c_{t_2}(n) - c_{t_1}(n) + 1}{t_1 \alpha(n) \left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2} \\
\leq &C \frac{(t_2 - t_1)\alpha(n)\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2}{t_1 \alpha(n)\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2}.
\end{aligned}
$$

By the Mean Value Theorem, we obtain

$$
t_2 - t_1 \leq r t_2^{1-\frac{1}{r}} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right).
$$

The assumptions about $c_{t_1}(n)$ and $c_{t_2}(n)$ imply that $t_2 - t_1 \geq \frac{1}{\alpha(n)\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2}$. So we arrive at

$$
\begin{aligned}
\sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} \leq &C \frac{t_2^{1-\frac{1}{r}}}{t_1}\left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right) \\
\leq &C \frac{t_1^{1-\frac{1}{r}}}{t_1}\left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right),
\end{aligned}
$$

where the second line applies

$$
t_2 = t_1 + t_2 - t_1 \leq t_1 + C \frac{1}{\alpha(n)\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2} \leq C t_1.
$$

The condition about r then entails

$$\begin{aligned}\sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} \leq & C t_1^{-\frac{1}{r}} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right) \\ \leq & C \left(\alpha(n) \left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2\right)^{\frac{1}{r}} \frac{\alpha(n)}{n} \frac{n}{\alpha(n)} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right) \\ \leq & C \frac{n}{\alpha(n)} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right)\end{aligned}$$

for some $C > 0$.
Recall that the second term is given by

$$\vartheta \frac{x_{n,\vartheta}^{c_{t_2}(n)-c_{t_1}(n)} - 1}{x_{n,\vartheta} - 1} \int_1^{x_{n,\vartheta}} v^{c_{t_1}(n)} \mathrm{d}v = \vartheta \frac{x_{n,\vartheta}^{c_{t_2}(n)-c_{t_1}(n)} - 1}{x_{n,\vartheta} - 1} \frac{x_{n,\vartheta}^{c_{t_1}(n)+1} - 1}{c_{t_1}(n) + 1}.$$

We have

$$\frac{x_{n,\vartheta}^{c_{t_2}(n)-c_{t_1}(n)} - 1}{x_{n,\vartheta} - 1} = \frac{\exp\left[\log(x_{n,\vartheta})(c_{t_2}(n) - c_{t_1}(n))\right] - 1}{x_{n,\vartheta} - 1}.$$

By assumption, $2 \leq c_{t_2}(n) - c_{t_1}(n) \leq 10$ and $t_2 \leq \left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2}$. Since $\log(x_{n,\vartheta}) \leq x_{n,\vartheta} - 1$, one obtains

$$\begin{aligned}\frac{x_{n,\vartheta}^{c_{t_2}(n)-c_{t_1}(n)} - 1}{x_{n,\vartheta} - 1} \leq & C \frac{\log(x_{n,\alpha})(c_{t_2}(n) - c_{t_1}(n) + 1)}{x_{n,\vartheta} - 1} \\ \leq & C \alpha(n) \left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2 (t_2 - t_1) \\ \leq & C \alpha(n) \left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2 t_2^{1-\frac{1}{r}} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right) \\ \leq & C \alpha(n) \left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{\frac{2}{r}} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right).\end{aligned} \tag{A.1.28}$$

We now distinguish two cases: If $c_{t_1}(n) \leq \frac{\alpha(n)}{2}$, we have

$$\begin{aligned}\int_1^{x_{n,\vartheta}} v^{c_{t_1}(n)} \mathrm{d}v \leq & (x_{n,\vartheta} - 1) x_{n,\vartheta}^{c_{t_1}(n)} \\ \leq & C \frac{\log\left(\frac{n}{\alpha(n)}\right)}{\alpha(n)} \exp\left(\frac{c_{t_1}(n)}{\alpha(n)} \log\left(\frac{n}{\alpha(n)\vartheta} \log\left(\frac{n}{\alpha(n)\vartheta}\right)\right)\right) \\ \leq & C \frac{\log\left(\frac{n}{\alpha(n)}\right)}{\alpha(n)} \left(\frac{n}{\alpha(n)} \log\left(\frac{n}{\alpha(n)}\right)\right)^{\frac{1}{2}}\end{aligned} \tag{A.1.29}$$

by Lemma 2.3.1 and Equation (A.1.22). Equations (A.1.28) and (A.1.29) taken together imply

$$\begin{aligned}\vartheta \frac{x_{n,\vartheta}^{c_{t_2}(n)-c_{t_1}(n)} - 1}{x_{n,\vartheta} - 1} \int_1^{x_{n,\vartheta}} v^{c_{t_1}(n)} \mathrm{d}v \leq & C \left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{\frac{3}{2}+\frac{2}{r}} \left(\frac{n}{\alpha(n)}\right)^{\frac{1}{2}} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right) \\ \leq & C \frac{n}{\alpha(n)} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right).\end{aligned}$$

If, on the other hand, $c_{t_1}(n) \geq \frac{\alpha(n)}{2}$,

$$\begin{aligned}
\frac{x_{n,\vartheta}^{c_{t_1}(n)+1} - 1}{c_{t_1}(n) + 1} \leq & \frac{2}{\alpha(n)} \left(x_{n,\vartheta}^{\alpha(n)} \right)^{\frac{c_{t_1}(n)+1}{\alpha(n)}} \\
\leq & \frac{4}{\alpha(n)} \exp \left(\frac{c_{t_1}(n) + 1}{\alpha(n)} \log \left(\frac{n}{\alpha(n)\vartheta} \log \left(\frac{n}{\alpha(n)\vartheta} \right) \right) \right) \\
\leq & C \frac{n}{(\alpha(n))^2} \log \left(\frac{n}{\alpha(n)} \right) \exp \left[-2 \frac{\log \log \left(\frac{n}{\alpha(n)} \right)}{\log \left(\frac{n}{\alpha(n)} \right)} \log \left(\frac{n}{\alpha(n)\vartheta} \log \left(\frac{n}{\alpha(n)\vartheta} \right) \right) \right] \\
\leq & C \frac{n}{(\alpha(n))^2} \left(\log \left(\frac{n}{\alpha(n)} \right) \right)^{-1} \qquad \text{(A.1.30)}
\end{aligned}$$

by Equations (A.1.22) and (A.1.23) as well as the definition of $c_t(n)$. Hence, by $r \geq 3$ and Equations (A.1.28) and (A.1.30),

$$\vartheta \frac{x_{n,\vartheta}^{c_{t_2}(n)-c_{t_1}(n)} - 1}{x_{n,\vartheta} - 1} \frac{x_{n,\vartheta}^{c_{t_1}(n)+1} - 1}{c_{t_1}(n) + 1} \leq C \frac{n}{\alpha(n)} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}} \right).$$

Step 2: Assume that $t_1, t_2 \geq \left(\log \left(\frac{n}{\alpha(n)} \right) \right)^{-2}$.
Then we have $c_{t_i}(n) = b_{t_i}(n) \geq \frac{\alpha(n)}{2}$ for $i = 1, 2$ by assumption and $b_t(n)$ is increasing in t, so

$$\begin{aligned}
\sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \leq & \frac{2\vartheta}{\alpha(n)} \sum_{j=b_{t_1}(n)+1}^{b_{t_2}(n)} x_{n,\vartheta}^j \\
= & \frac{2\vartheta}{\alpha(n)} x_{n,\vartheta}^{b_{t_1}(n)+1} \frac{x_{n,\vartheta}^{b_{t_2}(n)-b_{t_1}(n)} - 1}{x_{n,\vartheta} - 1} \\
\leq & C \frac{n}{(\alpha(n))^2} \log \left(\frac{n}{\alpha(n)} \right) t_1 \frac{\log(x_{n,\vartheta})(b_{t_2}(n) - b_{t_1}(n))}{x_{n,\vartheta} - 1}
\end{aligned}$$

by Equation (A.1.24) and the Mean Value Theorem. Hence,

$$\begin{aligned}
\sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \leq & C \frac{n}{(\alpha(n))^2} \log \left(\frac{n}{\alpha(n)} \right) t_1 (b_{t_2}(n) - b_{t_1}(n) + 1) \\
\leq & C \frac{n}{(\alpha(n))^2} \log \left(\frac{n}{\alpha(n)} \right) t_1 \frac{\alpha(n)}{\log \left(\frac{n}{\alpha(n)} \right)} (\log(t_2) - \log(t_1)) \\
\leq & C \frac{n}{\alpha(n)} t_1 \frac{1}{t_1} (t_2 - t_1) \\
\leq & C \frac{n}{\alpha(n)} \left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}} \right).
\end{aligned}$$

Step 3: Let $t_1 < \left(\log \left(\frac{n}{\alpha(n)} \right) \right)^{-2}$ and $t_2 > \left(\log \left(\frac{n}{\alpha(n)} \right) \right)^{-2}$.
Assume further that $c_{t_1}(n) \geq c\alpha(n)$ for some $c > 0$ since $c_{t_2}(n) = b_{t_2}(n) \geq \frac{\alpha(n)}{2}$, $c_{t_2}(n) - c_{t_1}(n) \leq 10$, and n is large. By suitably modifying the calculations performed in Step 2, we arrive at

$$\begin{aligned}
& \sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j} x_{n,\vartheta}^j \\
\leq & C \frac{n}{(\alpha(n))^2} \log \left(\frac{n}{\alpha(n)} \right) t_2 (c_{t_2}(n) - c_{t_1}(n) + 1) \\
\leq & C \frac{n}{(\alpha(n))^2 \log \left(\frac{n}{\alpha(n)} \right)} \left[b_{t_2}(n) - b_{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2}}(n) + c_{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2}}(n) - c_{t_1}(n) + 1 \right],
\end{aligned}$$

where the second line applies $t_2 = t_1 + t_2 - t_1 \leq C\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2}$, which is a consequence of $10 \geq c_{t_2}(n) - c_{t_1}(n) \geq C\alpha(n)\log\left(\frac{n}{\alpha(n)}\right)(t_2 - t_1)$ and Equation (A.1.25). A short calculation yields

$$\begin{aligned}
& b_{t_2}(n) - b_{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2}}(n) + c_{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2}}(n) - c_{t_1}(n) + 1 \\
\leq & C\left[\frac{\alpha(n)}{\log\left(\frac{n}{\alpha(n)}\right)}\left(\log(t_2) - \log\left(\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2}\right)\right)\right. \\
& \left. + \alpha(n)\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2\left(\frac{1}{\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2} - t_1\right)\right] \\
\leq & C\alpha(n)\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^2[t_2 - t_1].
\end{aligned}$$

Altogether, we have

$$\begin{aligned}
\sum_{j=c_{t_1}(n)+1}^{c_{t_2}(n)} \frac{\vartheta}{j}x_{n,\vartheta}^j \leq & C\frac{n}{\alpha(n)}\log\left(\frac{n}{\alpha(n)}\right)(t_2 - t_1) \\
\leq & C\frac{n}{\alpha(n)}\log\left(\frac{n}{\alpha(n)}\right)t_2^{1-\frac{1}{r}}\left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right) \\
\leq & C\frac{n}{\alpha(n)}\log\left(\frac{n}{\alpha(n)}\right)\left(\log\left(\frac{n}{\alpha(n)}\right)\right)^{-2+\frac{2}{r}}\left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right) \\
\leq & C\frac{n}{\alpha(n)}\left(t_2^{\frac{1}{r}} - t_1^{\frac{1}{r}}\right).
\end{aligned}$$

The claim is proved. □

A.2. Markov Chain Monte Carlo

The present section presents an algorithm with which it is possible to perform simulations of the model of random permutations without macroscopic cycles. Recall that the uniform and the Ewens measure are invariant distributions with respect to random transpositions which act as as split-merge process on the corresponding cycle structure (see Section 1.1.3). By not multiplying the underlying permutation with the random transposition if multiplication were to result in merging two cycles whose combined sizes exceed the maximal cycle length $\alpha(n)$, we arrive at a Markov chain whose state space is $S_{n,\alpha}$.
More precisely, let $g_s, g_m \in [0,1]$ such that $g_s/g_m = \vartheta$. Then P_α, defined by

$$P_\alpha(\sigma; \tau_{i_1 i_2} \circ \sigma) := \frac{1}{\frac{1}{2}n(n-1)} \begin{cases} g_s & \text{if } C(\tau_{i_1 i_2} \circ \sigma) = C(\sigma) + 1 \\ g_m & \text{if } C(\tau_{i_1 i_2} \circ \sigma) = C(\sigma) - 1 \text{ and } \tau_{i_1 i_2} \circ \sigma \in S_{n,\alpha} \\ 0 & \text{if } C(\tau_{i_1 i_2} \circ \sigma) = C(\sigma) - 1 \text{ and } \tau_{i_1 i_2} \circ \sigma \notin S_{n,\alpha} \end{cases}$$

and $P_\alpha(\sigma;\sigma) := 1 - \sum_{i_1 < i_2} P_\alpha(\sigma; \tau_{i_1 i_2} \circ \sigma)$, is a transition matrix on the state space $S_{n,\alpha}$ (cf. Equation (1.1.8)). Intuitively, as in Section 1.1.3, we first sample a transposition $\tau_{i_1 i_2}$ uniformly from the set of transpositions. Then, if multiplying σ with said transposition $\tau_{i_1 i_2}$ would split a cycle, this cycle is indeed split only with probability g_s. If multiplication of $\tau_{i_1 i_2}$ with σ were to merge two cycles, there are two cases to consider: If the length of the resulting cycle were to exceed $\alpha(n)$, the two cycles are not merged. Otherwise they are merged with probability g_m.
In a similar way as in Section 1.1.3, one easily sees that $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ satisfies the detailed balance condition for P_α, so convergence to equilibrium allows us to sample permutations which are approximately distributed according to $\mathbb{P}_{n,\alpha}^{(\vartheta)}$. The general approach of simulating a distribution by finding a Markov chain which converges to said distribution and then iterating the Markovian dynamics goes under the name of Markov chain Monte Carlo.
In principle, there are many methods to quantify the rate of convergence to equilibrium one might apply to the Markov chain defined by P_α. Since P_α is reversible, estimating the spectral gap (see, e.g., [**42**, Chapters 12 and 13]) or applying the theory of networks (cf., e.g., [**42**, Chapter 9]) are natural candidates for trying to do so. In the context of this thesis, however, simulations only served a heuristic purpose, so we did not estimate the rate of convergence. Instead we considered the behaviour of the total number of cycles and of the number of cycles of maximal length to obtain a heuristic value for the number of updates necessary for mixing. A *C#* implementation of the algorithm for $\vartheta = 1$, which has been utilized to produce the data presented in, e.g., Figures 2.7.1 and 2.7.2, can be found in Section A.2.1.

A.2.1. Implementation of Simulations. In this section we provide a listing of the implementation in C# of the algorithm described in Section A.2 for $\vartheta = 1$, $n = 10^6$, and $\alpha(n) = n^{\frac{1}{2}} = 1000$. For each realization, $5 \cdot 10^8$ iterations of the Markov chain are performed.

```
class Program
{
static void Main(string[] args)
{

uint TUpd = 0; //number of updates
int El1, El2, El3, Zyk1, Zyk2; //elements and cycle lengths
int SysGr, ZykL;
SysGr = 1000000; //system size
ZykL = 1000; //maximal cycle length
int AnzZyk = SysGr; //total number of cycles
int[] Struk = new int[ZykL]; //cycle structure
Random Rnd = new Random();
string zwischen = "";

int zaehl; //auxiliary variable for iteration

Struk[0] = SysGr; //initialisation as identity
for (int j = 1; j <= ZykL - 1; j++)
```

```
{
    Struk[j] = 0;
}

uint Laufzeit = 500000000; //number of iterations

zwischen = string.Format("{{"); //writing into files
File.WriteAllText(@"C:\Users\schäfer\Desktop\daten.txt", zwischen);
File.WriteAllText(@"C:\Users\schäfer\Desktop\zykelanzahl.txt", zwischen);

for (int n = 1; n <= 400; n++) //loop for number of samples
{
  for (int i = 1; i <= Laufzeit; i++) //loop for iterations of MC
  {
    zaehl = SysGr;
    TUpd++;

    El1 = Rnd.Next(1, SysGr + 1); //random no. from {1,2,...,SysGr}
    Zyk1 = ZykL;
    El2 = Rnd.Next(1, SysGr); //random no. from {1,2,...,SysGr-1}
    Zyk2 = ZykL;

    while (El1 <= zaehl) //finding length of cycle with El1
    {
      Zyk1--;
      zaehl = zaehl - Struk[Zyk1];
    }

    zaehl = SysGr - 1;

    while (El2 <= zaehl) //finding length of cycle with El2
    {
      Zyk2--;
      if (Zyk1 == Zyk2) { zaehl = zaehl - Struk[Zyk2] + 1; }
      else
      {
        zaehl = zaehl - Struk[Zyk2];
      }
    }

    if (Zyk1 == Zyk2) //cycles with El1 and El2 have same length
    {
      El3 = Rnd.Next(1, Struk[Zyk1]);
      if (El3 <= Zyk1) //El1 and El2 belong to the same cycle
      {
        Struk[El3 - 1] = Struk[El3 - 1] + El3;
        Struk[Zyk1 - El3] = Struk[Zyk1 - El3] + Zyk1 + 1 - El3;
        Struk[Zyk1] = Struk[Zyk1] - Zyk1 - 1;
        AnzZyk++;
      }
      else //El1 and El2 belong to different cycles
      {
        if (2 * Zyk1 + 2 <= ZykL) //merging possible
        {
          Struk[Zyk1] = Struk[Zyk1] - 2 * Zyk1 - 2;
          Struk[2 * Zyk1 + 1] = Struk[2 * Zyk1 + 1] + 2 * Zyk1 + 2;
```

```
          AnzZyk--;
        }
        else //merging not possible
        {
          TUpd--;
        }
      }
    }
    else //cycles with El1 and El2 have different lengths
    {
      if (Zyk1 + Zyk2 + 2 <= ZykL) //merging possible
      {
        Struk[Zyk1] = Struk[Zyk1] - Zyk1 - 1;
        Struk[Zyk2] = Struk[Zyk2] - Zyk2 - 1;
        Struk[Zyk1 + Zyk2 + 1] = Struk[Zyk1 + Zyk2 + 1] + Zyk1 + Zyk2 + 2;
        AnzZyk--;
      }
      else //merging not possible
      {
        TUpd--;
      }
    }
  }

  //writing data into a file

  zwischen = string.Format(",{0}", AnzZyk);
  File.AppendAllText(@"C:\Users\schäfer\Desktop\zykelanzahl.txt", zwischen);

  for (int m = 1; m <= ZykL - 1; m++)
  {
    zwischen = string.Format(",{0}", Struk[m]);
    File.AppendAllText(@"C:\Users\schäfer\Desktop\daten.txt", zwischen);
  }

  zwischen = string.Format("}},");
  File.AppendAllText(@"C:\Users\schäfer\Desktop\daten.txt", zwischen);
}

//outside of loop
zwischen = string.Format("}}");
File.AppendAllText(@"C:\Users\schäfer\Desktop\daten.txt", zwischen);
File.AppendAllText(@"C:\Users\schäfer\Desktop\zykelanzahl.txt", zwischen);
}
}
```

LISTING A.2.1

A.3. Notation and Conventions

- $a_n = \mathcal{O}(b_n)$ as $n \to \infty$ if there are constants $C, N > 0$ such that $|a_n| \leq C|b_n|$ for all $n \geq N$
- $f_n(t) = \mathcal{O}(g_n(t))$ pointwise in t as $n \to \infty$ if for each t there are constants $C_t, N_t > 0$ such that $|f_n(t)| \leq C_t |g_n(t)|$ for all $n \geq N_t$
- $f_n(t) = \mathcal{O}(g_n(t))$ uniformly in $t \in T_n$ as $n \to \infty$ if there are constants $C, N > 0$ such that $\sup_{t \in T_n} \frac{|f_n(t)|}{|g_n(t)|} \leq C$ for all $n \geq N$
- $a_n = o(b_n)$ as $n \to \infty$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$
- $f_n(t) = o(g_n(t))$ pointwise in t as $n \to \infty$ if for each t we have $\lim_{n\to\infty} \frac{f_n(t)}{g_n(t)} = 0$
- $f_n(t) = o(g_n(t))$ uniformly in $t \in T_n$ as $n \to \infty$ if $\sup_{t\in T_n} \left|\frac{f_n(t)}{g_n(t)}\right| \xrightarrow{n\to\infty} 0$
- $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$
- $f_n(t) \sim g_n(t)$ pointwise in t as $n \to \infty$ if for each t we have $\lim_{n\to\infty} \frac{f_n(t)}{g_n(t)} = 1$
- $f_n(t) \sim g_n(t)$ uniformly in $t \in T_n$ as $n \to \infty$ if $\sup_{t\in T_n} \left|\frac{f_n(t)}{g_n(t)} - 1\right| \xrightarrow{n\to\infty} 0$
- $a_n = \Omega(b_n)$ as $n \to \infty$ if there are constants $C, N > 0$ such that $|a_n| \geq C|b_n|$ for all $n \geq N$
- $B_r(x)$: the open ball of radius r about x in a metric space
- ∂B: the boundary of a set B in a topological space
- S_n: the symmetric group of permutations of n elements
- $S_{n,\alpha}$: the set of permutations of n elements whose cycles are not longer than $\alpha(n)$
- $\mathbb{P}$ and $\mathbb{E}$: a generic probability measure and the corresponding expectation
- $\mathbb{P}_n^{(\vartheta)}$ and $\mathbb{E}_n^{(\vartheta)}$: Ewens measure with parameter ϑ on S_n and the corresponding expectation
- $\mathbb{P}_{n,\alpha}^{(\vartheta)}$ and $\mathbb{E}_{n,\alpha}^{(\vartheta)}$: constrained Ewens measure with parameter ϑ and its expectation
- $C_j = C_j(\sigma)$: the number of cycles of length j of a permutation σ
- $C = C(\sigma)$: the total number of cycles of a permutation σ
- $\mathbb{N}$: the set of natural numbers excluding 0
- $\mathbb{N}_0$: the set of natural numbers including 0
- $\Im(z)$: the imaginary part of a complex number z
- $\Re(z)$: the real part of a complex number z
- $a \wedge b$: the minimum of two real numbers a and b
- $a \vee b$: the maximum of two real numbers a and b
- $(a)_+$: the positive part of a real number a
- $(a)_-$: the negative part of a real number a

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